02

AMERICAN
JOURNAL OF MATHEMATICS

FRANK MORLEY, EDITOR
COHEN, ASSOCIATE EDITOR
WITH THE COOPERATION OF
CHARLOTTE A. SCOTT, A. B. COBLE
AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XLVII

BALTIMORE: THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER, *New York*	WILLIAM WESLEY & SON, *London*
E. STEIGER & CO., *New York*	ARTHUR F. BIRD, *London*
G. E. STECHERT & CO., *New York*	A. HERMANN, *Paris*

1925

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

Cula - Ho2833-42-po147411

INDEX.

BELL, E. T. On Generalizations of the Bernoullian Functions and Numbers, ... 277

BUCHANAN, H. E. The Construction of Certain Periodic Orbits of the Three Body Problem, ... 289

DALE, JULIA. Some Properties of the Exponential Mean, . . . 71

DAVIS, H. T. An Extension of the Problem of the Elastic Bar, . 101

DINES, LLOYD L. On Certain Symmetric Sums of Determinants, . 249

FRÉCHET, MAURICE. L'expression la plus génerale de la "distance" sur une droite, ... 1

MERRIAM, GAYLORD M. On Certain Theorems Regarding Summable Series and their Application to the Double and Triple Fourier's Series, ... 125

MILLER, G. A. Imprimitive Substitution Groups, ... 176

MORLEY, FRANK. On an Equation of Planar Motion, ... 98

MORLEY, FRANK. The Eliminant of a Net of Curves, ... 91

MURRAY, F. H. Generalization of Certain Theorems of Bohl, . . 25

NYSWANDER, J. A. A Direct Solution of Systems of Linear Differential Equations having Constant Coefficients, ... 257

RAINICH, G. Y. Second Paper on Tensor Analysis, ... 11

RAINICH, G. Y. Third Paper on Tensor Analysis, ... 225

REINSCH, B. P. Expansion Problems in Connection with the Hypergeometric Differential Equation, ... 45

ROOS, C. F. A Mathematical Theory of Competition, ... 163

TORREY, MA[illegible] I. Classification of Monoidal Involutions having a Fixed Tangent Cone, ... 181

TURNER, B. M. A Configuration of Thirteen Pencils of Cubics and Cubics with Three Real Inflexions, ... 149

VANDIVER, H. S. On the Power Characters of Units in a Cyclotomic Field, ... 140

WEISNER, LOUIS. On the Sylow Subgroups of the Symmetric and Alternating Groups, ... 121

WINGER, R. M. Self-Projective Rational Septimics, ... 207

CAL
Sec. No.
Book No.
LIBRARY.

L'expression la plus générale de la "distance" sur une droite.

PAR MAURICE FRÉCHET.
(*Université de Strasbourg*)

Position du problème. Le problème que nous nous proposons de résoudre est le suivant:

Etant donnée une droite indéfinie, attacher à tout couple de points de cette droite, d'abscisses x, x' un nombre (x, x') qu'on pourra appeler distance généralisée de x et de x' satisfaisant aux conditions suivantes.

1° On a $(x, x') = (x', x) \geqq 0$.

2° Le nombre (x, x') est positif si x, x' sont distincts et nul s'ils coïncident.

3° Quels que soient les points x, x', x'', on a

$$(x', x'') \leqq (x, x') + (x, x'').$$

4° La condition nécessaire et suffisante pour qu'une suite de points $x_1, x_2, \cdots, x_n, \cdots$, convergent vers un point x est que la suite de nombres $(x, x_1), (x, x_2), \cdots, (x, x_n), \cdots$, converge vers zéro.

Un solution évidente du problème consiste à prendre

$$(x, x') = | x - x' | \times \text{constante}. \tag{1}$$

Mais nous nous proposons de *trouver l'expression la plus générale de la distance généralisée.*

Solutions particulières. Il est manifeste que la solution que nous venons d'obtenir n'est, en effet, pas la seule. Si par exemple on fait correspondre
droite x, x' biunivoquement et bicontinuement une courbe de Jordan
ts multiples (extrémités exclues, s'il en existe), la distance géo-
tre les deux points X', X'' de cette courbe qui correspondent aux
x', x'' de Δ, vérifiera les propriétés 1°, 2°, 3°, 4°. Par consé-

quent, on pourra prendre pour distance généralisée (x', x'') la distance géométrique de X' et X''.

Cette solution est essentiellement distincte de la solution (1), si la courbe de Jordan n'est pas une ligne droite; en effet dans ce cas, on aura pour trois valeurs x', x'', x''' de x correspondant à trois points non alignés X', X'', X'''.

$$| (x', x'') - (x', x''') | < (x'', x''') < (x', x'') + (x', x''')$$

alors que dans le cas de la solution (1) l'un des signes d'inégalité serai remplacé par un signe d'égalité.

Cette remarque montre en outre que contrairement à ce qu'il serait peu être naturel de supposer la solution générale ne s'obtient pas en transforman la droite Δ en elle même par une transformation biunivoque et bicontinue et prenant pour (x', x'') la distance géométrique des points correspondants à *x' et x''.*

Propriétés de la distance généralisée. Il ne sera pas inutile d'établir deux propriétés de la distance généralisée qui découlent naturellement de sa définition.

Tout d'abord la distance généralisée (x', x'') est bornée dans tout intervalle fini I. Car dans le cas contraire, on pourrait trouver dans I, x'_p, x''_p tels que

$$(x'_p,\ x''_p) > p.$$

Et on pourrait extraire de la suite des couples x'_p, x''_p une suite convergeant vers le couple t'_0, t''_0. On aurait donc d'après 3°, si après un changement de notation convenable, on suppose que ceci a lieu pour la suite donnée:

$$p < (x'_p\ x''_p) \leq (x'_p,\ t'_0) + (t'_0,\ t''_0) + (t''_0,\ x''_p)\,;$$

le second membre qui converge d'après 4° vers $(t'_0,\ t''_0)$ ne pourrait donc rester supérieur à p.

En outre la distance généralisée tend uniformément vers zéro dans I quand $| x' - x'' |$ tend vers zéro. Autrement il existerait un nombre $\epsilon > 0$ tel que quel que soit p, il existe un couple x'_p, x''_p tel que

$$| x'_p - x''_p | < 1/p, \quad (x'_p,\ x''_p) > \epsilon.$$

Comme précédemment on peut supposer que le couple x'_p, x''_p

(t'_0, t''_0) déterminée; mais cette fois on aura nécessairement $t'_0 = t''_0$. Alors:

$$0 < \epsilon < (x'_p, x''_p) \leqq (x'_p, t'_0) + (t''_0, x''_p),$$

et le second membre tendant vers zéro, ne pourrait rester supérieur à ϵ.

La distance généralisée (x', x'') est une fonction des deux variables x', x''. C'est une fonction uniformément continue de x', x'' dans tout intervalle I. Autrement dit, $\epsilon > 0$ etant donné, on peut trouver un nombre $\eta > 0$, tel que les inégalités $| x' - y' | < \eta$, $| x'' - y'' | < \eta$ vérifiées dans I entraînent

$$| (x', x'') - (y', y'') | < \epsilon.$$

En effet, on déduit de 3°

$$| (x', x'') - (y', y'') | \leqq (x', y') + (x'', y'')$$

et d'après la propriété précédemment établie, on peut trouver η tel que les inégalités

$$| x' - y' | < \eta, \quad | x'' - y'' | < \eta \qquad \text{vérifiées dans I}$$

entraînent

$$(x', y') < \eta/2, \quad (x'', y'') < \eta/2.$$

Recherche de la solution générale du problème. Nous avons vu qu'on pouvait former des solutions du problème au moyen de courbes de Jordan mises en correspondance avec Δ. Mais le raisonnement que nous avons fait garde sa valeur quel que soit le nombre de dimensions de l'espace dans lequel est tracé cette courbe.

Cette remarque nous conduit à chercher si l'on pourrait obtenir la solution du problème en utilisant une courbe de Jordan tracée dans un espace à une infinité de coordonnées, c'est à dire où chaque point X est défini par une suite infinie de nombres réels $X_1, X_2, \cdots, X_n, \cdots$, et où on a adopté un sens déterminé pour la notion de suite convergente de points $X^{(1)}, X^{(2)}, \cdots, X^{(p)}, \cdots$, de cet espace.

On sait que suivant les applications qu'on a en vue plusieurs définitions équivalentes des suites convergentes peuvent être envisagées. Notre choix
a sur la définition qui fournit l'espace D_ω, pour profiter d'une pro-
space D_ω qui nous sera utile,* et que nous démontrerons ici.

D_ω est un espace à une infinité de coordonnées où une suite

---

i *del Circole Matematico di Palermo*, t. 30, 1910, p. 12.

de points $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$, est dite converger vers un point X lorsque les coordonnées de $X^{(n)}$ tendent uniformément vers les coordonnées de même rang de X.

On peut définir dans cet espace une " distance " satisfaisant aux conditions 1°, 2°, 3°, 4°, en appelant distance de deux points X, X' de D_ω et en désignant par (X, X') la borne supérieure des valeurs absolues $| X_n - X'_n |$ des différences des coordonnées de même rang de X et de X'.

Pour éviter d'introduire des distances infinies actuelles, il sera commode de limiter D_ω aux points dont les coordonnées sont bornées (leurs bornes pouvant varier avec chaque point).

Solution générale du problème. L'expression (x, x') la plus générale de la " distance généralisée " de deux points x, x' d'une droite Δ s'obtient de la façon suivante: ou considére une courbe de Jordan sans points multiples de l'espace D_ω. Par définition, il existe une correspondance biunivoque et bicontinue entre C et la droite Δ; si X, X' sont les points de C qui correspondent aux points x, x' de Δ, on prendra pour (x, x'), la distance (X, X'), telle qu'elle a été definie plus haut.

1° Supposons donnée l'expression (x, x') de la distance généralisée sur la droite Δ. Nous allons définir la courbe C. Pour cela, suivant le procédé cité ci-dessus,* nous appellerons $t_1, t_2, \cdots, t_n, \cdots$, les abscisses d'un ensemble dénombrable partout dense de points de la droite Δ. (Par exemple, ce seront les abscisses rationnelles), et nous définirons C par les expressions suivantes de coordonnées $x_1, x_2, \cdots$, d'un point variable X de C:

$$(1)\quad X_1 = (x, t_1), \quad X_2 = (x, t_2) - (t_2, t_1), \cdots, \qquad X_n = (x, t_n) - (t_n, t_1), \cdots$$

A chaque point de coordonnée x de Δ correspond un point X de C, car non seulement les coordonnées de X sont numériquement déterminées, mais pour une valeur déterminée de x, ces coordonnées sont bornées, puisque d'après 3°, on a quelque soit n

$$| X_n | \leq (x, t_1)$$

La distance (X, X') de deux points X, X' de C est par déf borne supérieure de

$$| X_n - X'_n | = | (x, t_n) - (x', t_n) |.$$

---

* *Rendiconti del Circole Matematico di Palermo*, t. 30, 1910, p. 12

Or d'après 3°, le second membre est au plus égal à (x, x') qui est indépendant de n, donc

$$(X, X') \leq (x, x'). \tag{2}$$

Puisque l'ensemble $t_1, t_2, \cdots$, est dense sur Δ, on peut extraire de cet ensemble une suite $t_{q_1}, t_{q_2}, \cdots$, qui converge vers x':

D'après 3°

$$|(x, x') - (x', t_{q_i})| \leq (x, t_{q_i}) \leq (x, x') + (x', t_{q_i})$$

Lorsqu'on tient compte de 4° ces deux inégalités donnent à la limite:

$$(x, x') = \lim_{i \to \infty} (x, t_{q_i}).$$

L'égalité

$$|X_{q_i} - X'_{q_i}| = |(x, t_{q_i}) - (x', t_{q_i})|$$

donne donc

$$(x, x') = \lim_{i \to \infty} |X_{q_i} - X'_{q_i}|$$

et comme

$$(X, X') \geqq |X_q - X'_q|$$

quelque soit q on aura

$$(X, X') \geqq (x, x'), \tag{3}$$

De (2) et (3), on tire enfin:

$$(X, X') = (x, x'). \tag{4}$$

On en déduit d'abord qu'à tout point X de C ne correspond qu'un point x de Δ; car s'il en correspondait un autre $x' \neq x$, on pourrait supposer dans l'égalité (4) que X' est identique à X et d'après 1°, que le premier membre est nul et le second positif. Ainsi C n'a pas de points multiples, la correspondance entre C et Δ est biunivoque. Elle est aussi bicontinue. En effet si X' tend vers X, (X, X') tend vers zéro, donc aussi (x, x') et par suite, d'après 4° x' tend vers x. Réciproquement, si x' tend vers x, (x, x') d'après 4° tend vers zéro; donc aussi (X, X') et par suite X' tend vers X.

D'après cela la courbe C, représentée par les équations (1) est représentée paramétriquement sous la forme

$$x_1 = X_1(x), \quad x_2 = X_2(x), \cdots,$$

où $X_1(x), X_2(x), \cdots,$ sont des fonctions individuellement continues de x. En effet quel que soit n

$$| X_n(x) - X_n(x') | \leqq (X, X') = (x, x')$$

et si x' tend vers x, (x', x) tend vers zéro et $X_n(x')$ tend vers $X_n(x)$. On doit remarquer de plus que les fonctions $X_1(x), X_2(x), \cdots,$ sont bornées dans leur ensemble et également continues dans tout intervalle partiel I de la droite Δ. En effet

$$| X_n(x) | = | (x, t_n) - (t_n, t_1) | \leqq (x, t_1)$$

et si x reste dans un intervalle fini, t_1 restant fixe, on a vu que (x, t_1) est borné.

Les fonctions $X_n(x)$ sont aussi également continues dans I, car

$$| X_n(x) - X_n(x') | \leqq (X, X') = (x, x')$$

et nous avons vu que si $\epsilon > 0$ est donné, il existe un nombre $\eta > 0$, tel que dans I

$$\text{pour} \quad | x - x' | < \eta \quad \text{on ait} \quad (x, x') < \epsilon \quad \text{et par suite.}$$
$$| X_n(x) - X_n(x') | < \epsilon \qquad \text{quel que soit } n.$$

Enfin remarquons que les extrémités de C s'il y en a sont exclues de la correspondance. Par extrémité de C nous entendons tout point de l'espace D_ω qui serait point limite de points de C correspondant à des points $x_1, x_2, \cdots$, de Δ qui tendent vers $+\infty$ ou vers $-\infty$. Il s'agit de montrer que s'il existe un tel point limite, il ne coïncide pas en position avec un point $X(x)$ de C correspondant à une abscisse finie de x. En effet, on aurait $(x, x_n) = (X(x), X(x_n))$ et le second membre tendant vers zéro (x, x_n) tendrait vers zéro et par suite x_n tendrait vers un nombre fini x.

Ceci montre que si la définition de la distance (x, x') sur la droite Δ est compatible avec une généralisation du critère de convergence de Cauchy, la courbe C est illimitée dans les deux sens. Sans quoi, il y aurait une suite de points de Δ $x_1, x_2, \cdots, x_n, \cdots$, tendant vers $+\infty$ ou vers $-\infty$ telle que la suite des $X(x_n)$ vérifie la condition de convergence de Cauchy. Donc aussi la suite des x_n puisque $(X(x_n), X(x_{n+p})) = (x_n, x_{n+p})$. Et alors la suite des x_n devrait être convergente.

2° Réciproquement considérons une courbe C quelconque de l'espace D_ω qu'on peut mettre en correspondance biunivoque et bicontinue (extrémités de C, s'il y en a exclues) avec la droite Δ. Si x est l'abscisse du point de Δ qui correspond au point X de C, les coordonnées de X seront des fonctions uniformes de x.

$$X_1 = X_1(x), \quad X_2 = X_2(x), \cdots,$$

et l'ensemble de ces fonctions ne peut prendre deux ensembles identiques de valeurs pour deux valeurs différentes de x.

Ceci étant, posons

$$(x, x') = (X(x), X(x'))$$

et montrons qu'on peut considérer (x, x') comme une solution du problème, c'est à dire qu'elle satisfait aux conditions 1°, 2°, 3°, 4°.

Or cela résulte: d'une part de ce que des conditions semblables sont vérifiées pour deux points quelconques X, X' de la courbe sans points multiples, V (et même pour deux points quelconques de l'espace D_ω comme je l'ai montré précédemment); d'autre part de ce que la correspondance entre C et Δ est biunivoque et bicontinue.

Dans le cas où la courbe C est illimitée dans les deux sens, la distance (x, x') ainsi définie est compatible avec une généralisation du critère de convergence de Cauchy.

Forme analytique de la solution. On peut aussi donner à la solution générale que nous venons d'obtenir une forme purement analytique, débarassée d'une terminologie géométrique (qui a pourtant l'avantage de présenter l'énoncé sous une forme plus ramassée et plus intuitive).

Il suffit pour cela de chercher à quelles conditions doivent satisfaire les fonctions $X_1(x)$, $X_2(x)$, $\cdots$, pour qu'elles puissent représenter les coordonnées d'une courbe de Jordan (extrémités, s'il y en a, exclues) sans points multiples, dans l'espace D_ω.

Il faut d'abord que l'ensemble des fonctions $X_1(x)$, $X_2(x)$ $\cdots$, supposées uniformes ne puisse prendre le même ensemble de valeurs pour deux valeurs de x distinctes.

Pour que le point $X_1(x)$, $X_2(x)$, $\cdots$, appartienne à D_ω, il faut ensuite que les fonctions de cette suite soient bornées dans leur ensemble pour chaque valeur de x considérée isolément.

On peut même voir qu'elles sont bornées dans leur ensemble dans tout intervalle partiel I. Car dans le cas contraire il existerait quel que soit p, une valeur x_p dans I et une fonction X_{n_p}, telle que $| X_{n_p}(x_p) | > p$. On peut comme précédemment supposer que la suite des x_p converge vers un point t_0 de I. Or on a

$$\begin{aligned} p < | X_{n_p}(x_p) | &\leq | X_{n_p}(x_p) - X_{n_p}(t_0) | + | X_{n_p}(t_0) | \\ &\leq (X(x_p), X(t_0)) + | X_{n_p}(t_0) | ; \end{aligned}$$

et le dernier membre qui devrait ainsi croître au dela de toute limite est la somme de deux quantités l'une tendant vers zéro l'autre bornée quand p varie.

Enfin les fonctions $X_1(x)$, $X_2(x)$, $\cdots$, sont aussi également continues dans tout intervalle I; il suffit de montrer que quel que soit ϵ, il existe un nombre η tel que dans I l'inégalité

$$| x - x' | < \eta \text{ entraîne } (X(x), X(x')) < \epsilon$$

puisque $| X_n(x) - X_n(x') | \leq (X(x), X(x'))$ quelque soit n. Or dans le cas contraire il existerait quelque soit p un couple x_p, x'_p tel que

$$| x_p - x'_p | < 1/p \qquad (X(x_p), X(x'_p)) \geq p.$$

On peut supposer comme précédemment que la suite des x'_p converge vers un point t'_0 de I. Mais alors il en est de même des x_p et comme on a

$$p \leq (X(x_p), X(x'_p)) \leq (X(x_p), X(t'_0)) + (X(x'_p), X(t'_0)).$$

le second membre qui tend vers zéro devrait croître infiniment.

Réciproquement les trois propriétés des X_n que nous venons de reconnaître comme nécessaires sont aussi suffisantes.

Le fait que les $X_n(x)$ sont bornées dans leur ensemble pour chaque valeur de x, montre que $X_1(x)$, $X_2(x)$, $\cdots$, sont les coordonnées d'un point $X(x)$ bien déterminé de D_ω.

La première propriété assure ensuite la biunivocité de la correspondance entre x de Δ et $X(x)$ de C.

Quand à la bicontinuité, nous remarquons que les fonctions $X_n(x)$ étant également continues dans tout intervalle I, à tout nombre $\epsilon > 0$ correspond un nombre $\eta > 0$, tel que dans I l'inégalité $| x - x' | < \epsilon$ entraîne $| X_n(x) - X_n(x') | < \eta$ et aussi en prenant la borne supérieure du premier membre quand n varie

$$(X(x), X(x')) \leq \eta$$

Par conséquent quand on prend pour I un intervalle fixe de centre x et qu'on fait tendre x' vers x, $X(x')$ tendra vers $X(x)$.

En ce qui concerne la réciproque, il faut montrer que si $X(x_p)$ tend vers $X(x)$, x_p tend vers x. En effet supposons d'abord que tous les x_p restent dans un intervalle fini I. Alors si x' est la plus grande (ou la plus petite) des limites de x_p pour p infini, il y a un suite x_{p_1}, x_{p_2}, $\cdots$, extraite des x_p qui tend vers x'; par suite comme nous l'avons vu $X(x_{p_q})$ tend vers $X(x')$ et aussi par hypothèse vers $X(x)$; donc $x = x'$. Comme ceci s'applique à la fois à la plus grande et à la plus petite des limites, on voit finalement que la suite elle même des x_p est convergente et tend vers x.

Reste le cas où l'on pourrait extraire des x_p une suite x_{p_1}, x_{p_2}, $\cdots$, qui converge vers $+\infty$ ou vers $-\infty$, par exemple vers $+\infty$. Alors le point $X(x)$ qui serait le point limite dans D_ω des point $X(x_{p_1})$, $X(x_{p_2})$, $\cdots$, d'une part appartiendrait à la courbe C, d'autre part étant une extrémité de C ne lui appartiendrait pas.

Par conséquent, la solution générale du problème posé peut s'exprimer analytiquement de la façon suivante.

Pour définir une distance généralisée sur une droite indéfinie Δ, on peut toujours procéder ainsi: on appellera distance généralisée de deux points x, x' la borne supérieure quand n varie de

$$| X_n(x) - X_n(x') |$$

en désignant par $X_1(x)$, $X_2(x)$, $\cdots$, $X_n(x)$, $\cdots$, une suite de fonctions uniformes qui sont sur tout intervalle fini bornées dans leur ensemble et également continues et telles de plus 1° qu'elles ne prennent pas le même ensemble de valeurs pour deux valeurs de x distinctes 2° que s'il existe une suite de valeurs de x, x_1, x_2, $\cdots$, tendant vers $+\infty$ ou vers $-\infty$ sur lesquelles les fonctions $X_1(x)$, $X_2(x)$, $\cdots$, convergent uniformément, l'ensemble des valeurs limites ne puisse être pris en un point de Δ.

Remarques I. Une méthode tout à fait analogue réussirait sans doute si Δ au lieu d'être une droite était un plan ou l'espace euclidien à trois dimensions, etc. . . . .

II. Il serait intéressant de rechercher si dans ce qui précéde on pourrait remplacer l'espace D_ω par un espace plus simple. Il est facile de voir qu'on ne peut le remplacer par un espace à un nombre fini n de dimensions. Pour un tel espace en effet, il existe une relation entre les distances de $n+2$ points qui n'a pas lieu nécessairement pour un courbe quelconque tracée dans un espace à une dimension de plus. Mais peut-être pourrait-on remplacer D_ω qui n'est pas séparable par un espace (à une infinité de dimensions) qui serait séparable?

21 Août 1924.

Second Paper on Tensor Analysis.

BY G. Y. RAINICH.

In this paper which forms the continuation of a paper entitled "Two-Dimensional Tensor Analysis Without Coördinates" (THIS JOURNAL, Vol. XLVI, No. 2, April, 1924, pp. 71-94) we shall follow in the main the same course as in the paper mentioned above but instead of considering a 2-dimensional surface immersed in a 3-dimensional Euclidean space we shall consider an n-dimensional curved space Σ immersed in an N-dimensional Euclidean space. It will be convenient for us to continue throughout all these papers the numbering of the sections which we started in the first paper and we shall refer to the sections and formulas of the preceding paper without mentioning its title.

As in the case of the first paper, most of the results are known,* and the main purpose is to *simplify* the theory and to eliminate all that is unnecessary.

§ 14. *Generalization of preliminary notions.*

We shall use as an abbreviation for N-dimensional Euclidean space simply N-E.S., for n-dimensional space (which in general will be curved), we shall write n-S.; these will be only *abbreviations* and *not mathematical symbols;* 2-S. will be another *word* for surface, 3-E.S. another *word* for space (ordinary), 1-S. another word for curve etc. When we wish to refer to a particular space we use letters or letters with subscripts.

We shall have here to deal with points and vectors in a N-E.S., but the operations and notations remain the same as in a 3-E.S.: we use capital Roman letters for points, small Roman letters for vectors, which we consider as differences between points. The vectors are freely moveable in the space (unless the opposite is explicitly stated); there are therefore no new difficulties connected with addition and multiplication of vectors, because we always can bring their initial points together, and then both vectors are in the same plane.

We shall often have to use projections (i. e. orthogonal projections) of points and vectors on an n-E.S., and we shall use the following.

---

* Compare for bibliography the article by J. A. Schouten and D. J. Struik in the 46 Vol. of *Palermo Rendiconti*, p. 162.

Remark. If an r-E.S. lies entirely in an n-E.S. (of course, $r < n$) then instead of projecting a point or a vector on this r-E.S. we may first project it on the containing n-E.S. and after this project the projection on the r-E.S. considered.

The product of two vectors x and y may be considered as the product of the length of x by the length of the projection of y on the direction of x; it follows from this and the foregoing remark that

$$x.y = x.\bar{y} \tag{14.1}$$

where $\bar{y}$ is the projection of y on any E.S. which contains x.

The definition of a linear function as a function $\phi(x)$, which satisfies the equation $\phi(x+y) = \phi(x) + \phi(y)$, holds here; a tensor is again a multilinear function and a *flat* tensorfield is an immediate generalization of a plane tensorfield (§ 4): if a function $\phi(A, x, y, \cdots)$ with one point argument A, which is a point of an n-E.S., and several vector arguments $x, y, \cdots$ which are vectors of the same n-E.S. is linear with respect to the vector arguments, we call it a flat tensorfield.

In § 1 we considered an axis and a curve in a plane, or as we shall say now, a 1-E.S. and a 1-S. in a 2-E.S.; in § 2 we considered a plane Π and a surface, say Σ, in the ordinary space—or as we shall say now: a 2-E.S. and a 2-S. in a 3-E.S. We shall consider now an n-E.S. Π and an n-S. Σ in an N-E.S. (an n-dimensional flat, or Euclidean, space Π and an n-dimensional cuved space Σ in an N-dimensional Euclidean space). We shall consider together with the points of Π the points of Σ whose (orthogonal) projections they are; in the cases considered before, all the vectors along which the points are projected (the vectors which have the projections for their initial points and the projected points for their tips) are parallel to each other and their direction was therefore always known, it was *the* direction perpendicular to Π; it was therefore enough to give the length (and the sense) of the *projecting vector*. Things are different when $N - n > 1$—a good illustration is the case of a curve in ordinary space where $N = 3$, $n = 1$; let us take for Σ a helix and for Π its axis; to each point A of Π there corresponds a point of Σ whose projection is A, but the projecting vectors have not all the same directions: as A moves on the axis, the projecting vector rotates; i. e. changes its direction. It will not suffice to give the length of the projecting vector $\phi(A)$; we shall have to give its direction as well; i. e. we shall have to consider the whole vector. To give Σ we shall have to consider at each point A of Π the vector—we shall designate it $f(A)$—which leads from the point A to the point of Σ whose projection is A. If in the case of a surface we had designated the unit perpendicular vector to Π by p, what we now designate by $f(A)$ would be equal to $p.\phi(A)$. In the general case $f(A)$ answers every purpose which we required of $\phi(A)$ in the case of the surface. The function $f(A)$ can be said to give us a representation of Σ in the sense that the point of Σ whose projection on Π is A is here given by

$$A + f(A).$$

The n-E.S., tangent to Σ at that point (if it exists), will be given by

$$A + h + f(A) + f'(A, h),$$

where $f'(A, h)$ is the differential of $f(A)$, and may be found (if it exists) by aid of the formula

$$f'(A, h) = \lim_{\lambda \to 0} \frac{f(A + \lambda h) - f(A)}{\lambda},$$

which corresponds to 2.3; the only difference being that here $f'(A, h)$, $f(A + \lambda h)$, and $f(A)$ are vectors perpendicular to Π, whereas in 2.3 we considered only the length of these vectors, they having a fixed direction. As in that case, the differential $f'(A, h)$ is a linear function of h; but it is a vector function now and cannot be considered as a product.

§ 15. *The indicator.*

We suppose that the n-S. Σ, which we are considering has a definite tangent n-E.S. at each point, and we shall consider the vectors of this n-E.S. at the point A as vectors of the bundle A, which belongs to Σ; they will be denoted by x_A, y_A etc.

We agree, as before, to denote by B_A the projection of the point B of Σ on the n-E.S. which is tangent to Σ at the point A and put

$$B - B_A = f(B_A);$$

$f(B_A)$ is then the projecting vector considered in the preceding section, and

15.1 $$B_A + f(B_A)$$

is the representation of Σ, with the tangent n-E.S. at A playing the rôle of Π; and

15.2 $$B_A + h_A + f(B_A) + f'(B_A, h_A)$$

is the representation of the n-E.S. tangent at B. If we make $B = A$ we have, as in 6.2,

$$f(A) + f'(A, h_A) = 0$$

whence

15.3 $$f(A)=0, \qquad f'(A,h_A)=0,$$

We assume now that the second differential of $f(B_A)$ exists at the point A and denote it by $s(h_A,k_A)$:

15.4 $$s(h_A,k_A)=f''(A,h_A,k_A).$$

The so defined $s(h_A,k_A)$ is a bilinear function whose arguments are vectors of the bundle A and whose values are vectors perpendicular to the bundle A, or normal vectors at A. This corresponds not to what we called the indicator, viz., $\sigma(h_A,k_A)$ in the case $n=2$, $N=3$ but to $a.\sigma(h_A,k_A)$ with a the normal unit vector; of course, before it was enough to know $\sigma(h_A,k_A)$, i. e. the length of $s(h_A,k_A)$, because the direction was *the* direction of the normal. Here we have to consider $s(h_A,k_A)$, and we shall call *this* the indicator.* It is not to be confused with $s(h_A)$—one of the factors, into which $\sigma(h_A,k_A)$ could be split up—there is no such thing unless $N-n=1$.

Now, all we did for the point A, we can do for every point of Σ, and we have an indicator at every point.

We have thus a function which has a point argument A and two vector arguments h_A and k_A, upon each of which it depends linearly. Still it does not correspond to what we called a tensorfield, because the values of s instead of being numbers are normal vectors. We shall abstain from introducing a name for such a function, because the indicator will be the only function of this kind which we shall have to consider.

The function $f(B_A)$ is considered in a flat space—the tangent n-E.S.; its second differential is, therefore, symmetric in the independent differentials (comp. 4.3). The same applies to the indicator which, at each point, is a particular value of $f''(B_A,h_A,k_A)$. We have, thus,

15.5 $$s(h_A,k_A)=s(k_A,h_A).$$

§ 16. *Another way of introducing the indicator. Inclination.*

The aim of this section is to prove a formula which will be a generalization of 7.5; but it is obvious that we shall have to change 7.5 considerably: in fact, the left-hand side of 7.5 cannot be generalized immediately, because there is no unique normal vector at each point; the right-hand side also would give us some difficulty; we should not be surprised therefore if the formula at which we arrive here will look altogether different from 7.5.

---

* The Vector Second Form which Wilson and Moore consider in the case $n=2$, $N>3$ is closely related to the so defined indicator. See § 25 of their paper in the 52 vol. of the *Proc. of the Amer. Acad. of Arts and Sciences.*

We introduced in § 9 the vector $x_{A \cdot B}$ as a vector of the bundle B, whose projection on the bundle A is the vector x_A. We shall adopt the same notation here.

Let us consider the expression 15.1 and 15.2 which give us the point B of Σ and the point of the n-E.S. tangent at B whose projection on the n-E.S. tangent at A is $B_A + h_A$. The difference between these two expressions

$$h_A + f'(B_A, h_A)$$

is a vector of the bundle B, whose projection on the bundle A is h_A; according to the notation introduced in this section we may therefore write

$$h_{A \cdot B} - h_A = f'(B_A, h_A). \tag{16.1}$$

Making $B_A = A + \lambda k_A$, and remembering that $f'(A, h_A) = 0$ (15.3) we have

$$\lim_{\lambda \to 0} \frac{h_{A \cdot B} - h_A}{\lambda} = f''(A; h_A, k_A) = s(h_A, k_A). \tag{16.2}$$

This formula gives us another definition of the indicator (and stands here for 7.5; it may also be compared with 7.31 and 10.1). But if we had first introduced the indicator in this way, we would have to prove its symmetry separately (given by 15.5).

The formula 16.2 is very important as it is, but we can replace it by a more general formula, removing several restrictions to which the left-hand side is subjected in its present form.

1. We defined $h_{A \cdot B}$ by its property of having h_A for its projection on the n-E.S. tangent at A; we can therefore say that the difference $h_{A \cdot B} - h_A$ is normal to Σ at A. But it is evident that it is not necessary for the validity of 16.2 to have this difference normal to A; if we would know about a vector h_B only that the ratio

$$\frac{h_B - h_A}{\lambda} \tag{16.3}$$

tends toward a vector normal to Σ at A we, evidently would have the same result.

2. Moreover, it is not necessary to have $B_A = A + \lambda k_A$; all we need is

$$\lim_{\lambda \to 0} \frac{B - A}{\lambda} = k_A \tag{16.4}$$

3. Finally, instead of considering 16.3 we may consider

$$\lim_{B\to A} \frac{h_B - h_A}{|B - A|},$$

where $|B - A|$ denotes the length of the vector $B - A$; this differs from the limit of 16.3 only by a constant factor, viz. $|k_A|$, when 16.4 holds.

We shall now sum up these remarks in the following form.

Definition. A variable vector h_B is said to *incline* toward a constant vector h_A if

$$\lim_{B\to A} \frac{h_B - h_A}{|B - A|}$$

is normal to Σ at the point A.

Theorem. If $\dfrac{B - A}{\lambda} \to k_A$ when $\lambda \to 0$ and h_B inclines toward h_A

16.5 $$\lim_{\lambda\to 0} \frac{h_B - h_A}{\lambda} = s(h_A, k_A)$$

where $s(h_A, k_A)$ is the indicator.

We have thus a new process of "inclination" which, as might be seen from § 9, and will be also seen later, plays a fundamental rôle in differentiation. This process of inclination is analogous in some respects to the process of tending toward a limit. We also have here a constant vector h_A and a variable vector h_B (belonging to a variable bundle); if we have two vectors h_B and k_B which incline toward h_A and k_A respectively, their sum $h_B + k_B$ inclines toward $h_A + k_A$ etc.

The vector $h_{A\cdot B}$ introduced in the beginning of this section furnishes us with an illustration of a vector which inclines toward h_A; another vector which inclines toward A is $h_{A.B}$ with the dot below: *we define it as the projection of* h_A on the bundle B.

We notice for further use that the relations 9.6 and 9.7 hold also in the general case; 9.7 is an immediate consequence of the definitions of $x_{A\cdot B}$ and $x_{A.B}$ and the proof of 9.6 is also easy.

Formula 16.5 contains in a nut-shell the whole general theory of curved spaces as far as second order properties are concerned.

Remark. We shall have to use in § 20 a still more general form of it: instead of considering in the left-hand side the constant vector h_A we may introduce there a variable vector $\bar{h}_A$ of the bundle A having h_A for its limit.

§ 17. *Internal and external properties of a curved space.*

The Riemann Tensor.

The indicator is essentially something external with respect to the space Σ, since it is a function whose values are *normal* vectors. The tangential vectors and inclination may, on the contrary, be interpreted in terms of the space itself, or in terms accessible to the inhabitants of the space. As concerns, first, vectors, formula 16.4 shows how a tangential vector can be introduced with the aid of a fixed point A, a variable point B and a variable number λ. This does not seem to be a very simple way of introducing vectors, but it is, as far as I can see, the simplest (if one does not introduce geodesics first; but the notion of geodesics is evidently more complicated than that of vectors) and appears to be quite natural after one thinks of it for a while. Once the vectors are introduced, inclination, which is a relation amongst vectors, also can be introduced—though not defined (except by its properties), because the definition of inclination involves subtraction of vectors belonging to two different bundles which operation cannot be performed by the inhabitants of Σ; given a variable vector x_B, they cannot decide by themselves whether it inclines toward x_A unless they are told; but we assume that inclination *is* introduced; i. e. the inhabitants *are* told of each variable vector whether it inclines toward a fixed vector.

We consider thus as internal the *vectors* of different bundles, the *operations* (addition, multiplication) on vectors belonging to the same bundle, and the notion of *inclination* (inclination is not independent of addition; but we shall not insist on this point here). All the properties which can be expressed in terms of these notions we shall call internal properties—all the others—external.

The question arises now whether all the properties of the indicator are external properties or whether we can learn something about it from the internal properties—from the inclination. We shall see in fact that some of the properties of the indicator are internal properties.

We start with the identity

$$17.1 \qquad x_{A\cdot B.C}\cdot y_{A\cdot C} - x_{A\cdot C.B}\cdot y_{A\cdot B} = \begin{vmatrix} x_{A\cdot B} - x_A & y_{A\cdot B} - y_A \\ y_{A\cdot C} - y_A & y_{A\cdot C} - y_A \end{vmatrix} .$$

Since $x_{A\cdot B.C}$ is the projection of $x_{A\cdot B}$ on the n-E.S. tangent to Σ at C which contains $y_{A\cdot C}$, we may apply 14.1 and write $x_{A\cdot B}\cdot y_{A\cdot C}$ instead of the first term on the left-hand side. Treating in the same way the second term and writing out the determinant we may write the identity to be proved as

$$(x_{A\cdot B} - x_A)(y_{A\cdot C} - y_A) - (x_{A\cdot C} - x_A)(y_{A\cdot B} - y_A) \\ - x_{A\cdot B}y_{A\cdot C} + x_{A\cdot C}y_{A\cdot B} = 0.$$

But instead of this we can write

$$-(x_{A\cdot B} - x_A)\,.\,y_A + (x_{A\cdot C} - x_A)\,.\,y_A - (y_{A\cdot C} - y_A)\,.\,x_A \\ + (y_{A\cdot B} - y_A)\,.\,x_A = 0;$$

and this is true because in each of the products the first factor is perpendicular to the bundle A (compare 15.1) and, therefore, to the second factor.

If now we make

$$\lim_{\lambda\to 0} \frac{B - A}{\lambda} = h_A \qquad \lim_{\mu\to 0} \frac{C - A}{\mu} = k_A$$

and take account of 17.1 and 16.2 we find

$$17.2 \qquad \lim_{\substack{\lambda\to 0\\ \mu\to 0}} \frac{x_{A\cdot B.C}\cdot y_{A\cdot C} - x_{A\cdot C.B}\cdot y_{A.B}}{\lambda\mu} = \begin{vmatrix} s(x_A, h_A) & s(x_A, k_A) \\ s(y_A, h_A) & s(y_A, k_A) \end{vmatrix}.$$

The left-hand side is an expression which does not involve operations on vectors belonging to different bundles. It is connected therefore with inner properties of the space, and the right-hand side involves the indicator; more precisely, if we know the indicator we can calculate the right-hand side but not vice versa, of course; the determinant of the right-hand side reflects only *some* of the properties of the indicator—we can change the indicator to a certain degree without changing this determinant—and we see that *these* properties of the indicator are internal properties of the space.

Just as in the preceding section, it is possible to generalize the formula 17.2 by removing some restrictions from the left-hand side; viz., we write our formula as

$$17.3 \qquad \lim_{\substack{\lambda\to 0\\ \mu\to 0}} \frac{x_{ABC}\cdot y_{AC} - x_{ACB}\cdot y_{AB}}{\lambda\mu} = \begin{vmatrix} s(x_A, h_A) & s(x_A, k_A) \\ s(y_A, h_A) & s(y_A, k_A) \end{vmatrix}.$$

Here x_{AB} denotes a vector,—function of x_A and B, which inclines toward x_A; x_{ABC} a vector, function of x_{AB} and C, which inclines toward x_{AC} etc. It is thus enough only to know the inclination in order to be able to compute the determinant of the right-hand side. This determinant is a function which has numerical values and which depends linearly on four vector arguments x, y, h, k. It is therefore a tensor, and this is precisely the Riemann Tensor of Σ at the point A. We denote it by $\rho(x, y; h, k)$. Its fundamental properties, i. e. the fact that it changes its sign when we interchange x and y, or h and k, the fact that it does not change when we interchange the couples

x, y and h, k and the relation

$$\rho(x, y; h, k) + \rho(x, h; k, y) + \rho(x, k; y, h) = 0,$$

are immediate consequences of its form 17.3 and of the symmetry of the indicator (15.5).

§ 18. *Tensorfields of a curved space and their differentiation.*

As in § 5, we consider a tensor, whose arguments are vectors of a bundle tangent to Σ, as a tensor belonging to Σ; and a *tensorfield* belonging to Σ will be constituted by tensors given in each bundle of Σ, or, what is the same thing, by a function $\phi(A, x_A, y_A, \cdots)$ with one point argument, which is a point of Σ, and several vector arguments, which are vectors of the corresponding bundle and upon which the function depends linearly.

As an example we may again consider the "fundamental" tensorfield

$$\epsilon(A, x_A, y_A) = x_A \cdot y_A, \tag{18.1}$$

which, however, is inconspicuous in the theory unless we introduce the usual coördinate representation of curved space. As another example may serve the tensorfield constituted by the Riemann Tensors at the points of Σ. The indicator does *not* constitute a tensorfield here, since its values are not numbers.

Using the notion of *inclination*, we can define the differential of a tensorfield $\phi(A, x_A, y_A, \cdots)$, in accordance with § 9, by

$$\phi'(A, x_A, y_A, \cdots; h_A) = \lim_{\lambda \to 0} \frac{\phi(B, x_B, y_B, \cdots) - \phi(A, x_A, y_A, \cdots)}{\lambda}, \tag{18.2}$$

with $\dfrac{B - A}{\lambda} \to h_A$, and $x_B, y_B, \cdots$ being vectors which *incline* toward $x_A, y_A, \cdots$, respectively. In particular we can put $x_B = x_{A{}^{\cdot}B}$ or $x_B = x_{A.B}$ etc.

If we consider again the fundamental tensor 18.1, we can prove that its differential is zero. In fact

$$\epsilon'(x_A, y_A, h_A) = \lim_{\lambda \to 0} \frac{x_{AB} \cdot y_{AB} - x_A \cdot y_A}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{x_B - x_A}{\lambda} \cdot y_B - \lim_{\lambda \to 0} x_A \cdot \frac{y_A - y_B}{\lambda},$$

and this is zero because $\lim \dfrac{x_B - x_A}{\lambda}$ and $\lim \dfrac{y_B - y_A}{\lambda}$ are normal vectors according to the definition of inclination, and are, therefore, perpendicular to x_A and $\lim y_B$.

We pass now to the generalization of the Codazzi equations (§ 10). We remember that the *indicator* is equal to the *second differential* of $f(B_A)$ at the point A; the relation 10.6 from which we deduced the Codazzi equations 10.7 in the case $n = 2$ can be said to express a connection between the *differential of the indicator* (which is connected with the space Σ, so that the differential is taken "along" Σ) and the *third differential of* $f(B_A)$ (which is taken along the tangent n-E.S.). We proceed now to find out how s' and f''' are connected in the general case.

We make $B_A = A + \mu l_A$, $C_A = B_A + \lambda k_A$, and have

$$\lim_{\lambda\to 0}\frac{C - B}{\lambda} = k_{A\cdot B},$$

as in § 10.

According to 16.1, we have

$$h_{A\cdot B} - h_A = f'(B_A, h_A), \qquad h_{A\cdot C} - h_A = f'(C_A, h_A);$$

so that

$$f''(B_A, h_A, k_A) = \lim_{\lambda\to 0}\frac{h_{A\cdot C} - h_{A\cdot B}}{\lambda}.$$

On the other hand, according to 16.2;

$$s(h_{A\cdot B}, k_{A\cdot B}) = \lim_{\lambda\to 0}\frac{h_{A\cdot B\cdot C} - h_{A\cdot B}}{\lambda},$$

Since, for $B = A$, the values of s and f'' coincide (by definition) we have

18.3 $$s'(h_A, k_A, l_A) - f'''(A_A, h_A, k_A, l_A) = \lim_{\substack{\lambda\to 0\\ \mu\to 0}}\frac{h_{A\cdot B\cdot C} - h_{A\cdot C}}{\lambda\mu}.$$

From this it follows at once that *the normal component of s' is equal to f'''*, which is the sought-for generalization of 10.6; and from this we arrive at the consequence, which may be looked upon as the generalization of the Codazzi relations, that

18.4 *the normal component of the differential of the indicator is symmetrical in its three arguments.*

Incidentally, if we want to consider the tangential component of the difference 18.3 we write the right-hand side with the dots below (which does not change its value) and multiply it by an arbitrary vector z_A. We have $h_{A.B.C}\cdot z_A = h_{A.B}z_{A.C}$, $h_{A.C}\cdot z_A = h_A z_{A.C}$, and the difference takes the form

$$\lim_{\substack{\lambda\to 0\\ \mu\to 0}}\frac{h_{A.B} - h_A}{\mu\lambda}\,.\,z_{A.C}.$$

We first pass to the limit $\mu \to 0$ and obtain

$$\lim_{\lambda \to 0} \frac{s(h_A, l_A) . z_{A.\sigma}}{\lambda} .$$

Since $s(h_A, l_A)$ is perpendicular to z_A, we write this

$$\lim_{\lambda \to 0} \left(s(h_A, l_A) . \frac{z_{A.\sigma} - z_A}{\lambda} \right) ,$$

which gives $s(h_A, l_A) . s(z_A', k_A)$. We thus have, dropping the subscript A,

$$z . s'(h, k, l) = s(h, l) . s(z, k) .$$

If we interchange k and l and take the difference, we obtain:

$$z . s'(h, k, l) - z . s'(h, l, k) = \begin{matrix} s(h, l) & s(h, k) \\ s(z, l) & s(z, k) \end{matrix} = \rho(h, z; l, k) .$$

The situation is then, the tangential component of the differential of the indicator is not, in general, symmetrical in its arguments: if we interchange the two last arguments, the difference is the Riemann Tensor.

The consideration of the tangential component of s' is, however, not necessary for our purpose, which is now the consideration of the differential of the Riemann Tensor. From its definition we have

$$\rho'(x, y; h, k, l) = \begin{vmatrix} s'(x, h, l) & s'(x, k, l) \\ s(y, h) & s(y, k) \end{vmatrix} + \begin{vmatrix} s(x, h) & s(x, k) \\ s'(y, h, l) & s'(y, k, l) \end{vmatrix} ;$$

since s' is, in each instance, multiplied by s, which is a normal vector, we need not take into account the tangential component, i. e. we can substitute for s' its normal component which, according to the generalization of the Codazzi relations 18.4, is symmetrical in its arguments. This leads to an immediate proof of the Bianchi relations 12.5 for the general case.

Tensorfields of Σ, their differentiation, and the Bianchi identities are internal things; but here they were discussed with the aid of external things like the indicator, its differential, and the Codazzi relations.

§ 19. *An r-S. in an n-S.*

In this section and the following we shall consider the generalizations of the developments of §§ 8 and 11; instead of a curve on a surface which is a 1-S. in a 2-S. we shall consider an r-S., which we shall call Σ_r, in an n-S. (with $r < n$), which we shall call Σ_n.

The question which we shall try to answer first is: what relation exists between the indicators $s_r(h_A, k_A)$ of Σ_r and $s_n(h_A, k_A)$ of Σ_n in a point A which belongs to both? First, we notice that the tangent r-E.S. of Σ_r at that point lies entirely in the n-E.S. tangent to Σ_n; we therefore shall consider in $s_n(h_A, k_A)$ only those values of h_A, k_A which belong to the tangent r-E.S. and so have the functions s_r and s_n for the same arguments.

We now introduce the formulas

19.1 $$s_n(h_A, k_A) = \lim_{\lambda \to 0} \frac{h_{AnB} - h_A}{\lambda}$$

19.2 $$s_r(h_A, k_A) = \lim_{\lambda \to 0} \frac{h_{ArB} - h_A}{\lambda}$$

where h_{AnB} and h_{ArB} denote vectors of the bundles B which incline toward h_A with respect to Σ_n and to Σ_r respectively; h_{AnB} may be the projection of h_A on the n-E.S. tangent to Σ_n at B, and h_{ArB} the projection of h_A on the r-E.S. tangent to Σ_r at the same point. Since this r-E.S. forms part of that n-E.S. the difference between these two vectors will be a vector of the n-E.S. tangent at B and perpendicular to the r-E.S. tangent at B; the difference between the limits will be a vector of the n-E.S. tangent at A and perpendicular to the r-E.S. tangent to A; we can say

The values of $s_n(h_A, k_A)$ for the arguments h_A and k_A for which $s_r(h_A, k_A)$ are determined (i. e. for h_A, k_A belonging to the r-E.S. tangent at A) *are projections of the corresponding values of $s_r(h_A, k_A)$.*

The formula 8.4 which contains the Euler and the Meusnier theorems may be interpreted as a special case of this. In fact, if we make $n = 2$, we have

19.3 $$s_2(h_A, k_A) = a . \sigma(h_A, k_A) .$$

The only vectors for which $s_1(h_A, k_A)$ is determined are vectors of the tangent to our curve; since they all have the same direction, there is no loss of generality if we put $h_A = k_A = t$, where t is the unit vector of that direction. It is clear that, in this case, we have $s_1(t, t) = \dot{t}$, and the underlined proposition above takes the form

$$a(a\dot{t}) = a . \sigma(t, t)$$

or, if we take 8.1 into consideration

$$a \frac{(an)}{\rho} = a . \sigma(t, t) .$$

Incidentally, we see what the meaning of the indicator for a 1-S., i. e. for a curve, is. In this case all the values of the indicator differ only by numerical factors of its value for unit vectors as arguments, and this last value is the curvature vector, or a vector which has the direction of the first normal and whose length is equal to the curvature of the curve. We may remark that our general point of view means a simplification even in the simplest case: the curvature is introduced without the notion of length of arc.

§ 20. *The geodesic indicator.*

We shall now generalize the operation considered in § 11. We again consider Σ_r in Σ_n. We had (19.2) for the inclination in Σ_r

$$\lim_{\lambda\to 0} \frac{h_{ArB}-h_A}{\lambda} = s_r(h_A, k_A).$$

The subtraction considered here cannot be performed in Σ_r, and is supposed to be performed in the containing N-dimensional Euclidean Space. But if we consider the n-S. Σ_n, of which Σ_r is a part and do not wish to step out of this Σ_n then, in analogy to 11.2 we shall consider

$$\lim \frac{h_{ArBnA}-h_A}{\lambda}$$

as the indicator of Σ_r in Σ_n, or as the *geodesic indicator* of Σ_r in Σ_n. But we have

$$\lim \frac{h_{ArBnA}-h_A}{\lambda} = \lim \frac{h_{ArBnA}-h_{ArB}+h_{ArB}-h_A}{\lambda}$$

$$= \lim_{\lambda\to 0} \frac{h_{ArB}-h_A}{\lambda} - \lim_{\lambda\to 0} \frac{h_{ArB}-h_{ArBnA}}{\lambda}$$

The minuend of the last expression is the indicator of Σ_r, and the subtrahend—as the remark at the end of § 16 permits to conclude—is the indicator of Σ_n. Hence, the geodesic indicator of Σ_r in Σ_n is the difference between the indicator of Σ_r (in the containing N-dimensional Euclidean Space) and the indicator of Σ_n.

Or, we may say also: if the indicator of Σ_r is decomposed into its tangential and normal components with respect to Σ_n, in which Σ_r lies, the tangential component is the geodesic indicator of Σ_r in Σ_n, and the values of the normal component are equal to the corresponding values of the indicator of Σ_n.

The results of § 11 are easily seen to follow from this as a special case.

These considerations may easily be extended. If Σ_t lies in Σ_r, Σ_r in Σ_n, the geodesic indicator of Σ_t with respect to Σ_n is equal to the geodesic indicator of Σ_t with respect to Σ_r plus the geodesic indicator of Σ_r with respect to Σ_n.

AUGUST 18, 1924.

ADDED NOV. 15, 1924.

There exists a more immediate generalization of the formula 7.5 of the first paper than that given by 16.2. In § 7 we considered at every point B, given by $B_A = A + \lambda k_A$, *the* unit normal vector b, a being the unit normal vector at A. We cannot speak here of *the* unit normal vector; but we shall consider at each point B some definite normal vector b, a being the vector which corresponds to $\lambda = 0$. The

$$\lim \frac{b - a}{\lambda},$$

when this limit exists, is not, in general, a vector of the tangent bundle A; we therefore consider the tangential component of $\lim \frac{b - a}{\lambda}$; this tangential component is a vector which depends only on a and k_A. We shall denote it by $\mathcal{F}(a, k_A)$. It can be shown that $\mathcal{F}$ depends on each argument linearly, and that the connection of this function $\mathcal{F}(a, k_A)$ with the indicator $\overset{0}{s}(h, k)$ of § 15 is given by the relation

$$h_A . \ \mathcal{F}(a, k_A) = a . s(h, k).$$

Here h_A and k_A are two vectors of the tangent bundle at A, and a a vector of the normal bundle.

Errata in the first paper:

P. 83, line 2 from above: read k_A instead of h_A.
P. 86, line 1 from above: read $x_{A \cdot B}$ instead of $x_{B \cdot A}$.
P. 88, line 6 from above: read $C_A = B_A + \lambda k_A$ instead of $C_A = B + \lambda k_A$.
P. 89, line 9 from above: read δ'' instead of δ.
P. 92, line 9 from above, read $x_{A.B.C.A} - x_{A.C.B.A}$ instead of $x_{A.B.C.A} - x_{A.B.C.A}$

Generalization of Certain Theorems of Bohl.

By F. H. Murray.

It is the object of this paper to generalize a certain number of theorems of P. Bohl [*] concerning the trajectories of a mechanical system in the neighborhood of a point of equilibrium. The coördinates of the system considered by Bohl satisfied equations of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega'_i}\right)-\frac{\partial T}{\partial \omega_i}=\frac{\partial U}{\partial \omega_i}, \qquad i=1, \cdots, n,$$

in which T is a definite quadratic form in the velocities, and the terms of the second order in U are a non-singular quadratic form. Under these and certain other assumptions the existence is established of trajectories which remain in a given neighborhood of the point of equilibrium for $t \geqq t_0$, for $t \leqq t_0$, or for all values of t; and the manifold of each type of trajectory is determined.

In the present paper a number of similar theorems are obtained concerning the trajectories in the neighborhood of a point solution of the equations of Hamilton

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}, \qquad i=1, \cdots, n,$$

in which F does not contain the time. A certain number of inequalities are also obtained for trajectories in the neighborhood of a periodic orbit, when F contains t explicitly. While it seems probable that the results obtained here can be still further generalized, they are sufficient for a number of applications which will be made in a following paper.

Although the theorems obtained include those of Bohl as a special case, no claim to originality of treatment is made, since his methods, with slight changes, are sufficient for this more general problem. A new difficulty which occurs here is that of determining the form of F after certain transformations; this is overcome by a method similar to one employed by Poincaré.[†]

---

[*] P. Bohl, "Über die Bewegung eines mechanischen Systems," *Journal für die reine und angewandte Mathematik*, Band 127 (1904), pp. 179-276.

[†] *Les méthodes nouvelles de la mécanique céleste*, t. III, § 319.

PART I. *Transformation of the equations of motion.*

1. The equations of variation.

Suppose the motion of the system defined by the canonical equations

$$(1) \qquad \frac{d\bar{x}_i}{dt} = \frac{\partial F}{\partial \bar{y}_i}, \quad \frac{d\bar{y}_i}{dt} = -\frac{\partial F}{\partial \bar{x}_i}, \qquad i = 1, \cdots, n,$$

and let a periodic solution be represented by the functions

$$(2) \qquad \bar{x}_i = \phi_i(t), \quad \bar{y}_i = \psi_i(t) .$$

Suppose the region (D) to consist of all points $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n, \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n)$ which satisfy the inequalities

$$(D) \qquad |\,\bar{x}_i - \phi_i(t)\,| < d', \quad |\,\bar{y}_i - \psi_i(t)\,| < d', \qquad i = 1, \cdots, n$$

for some value of t; it will be assumed that the function F and its partial derivatives of the first and second orders are continuous within this region. If the transformation $\bar{x}_i = \phi_i + \xi_i$, $\bar{y}_i = \psi_i + \eta_i$ is made, from (1) we obtain the equations

$$(3) \qquad d\xi_i/dt = [\partial F/\partial \bar{y}] - [\partial F/\partial \bar{y}_i]^0, \; d\eta_i/dt = -[\partial F/\partial \bar{x}_i] + [\partial F/\partial \bar{x}_i]^0$$

or if

$$\bar{F} = [F] - \sum_{i=1}^{n} \left([\partial F/\partial \bar{x}_i]^0 \xi_i + [\partial F/\partial \bar{y}_i]^0 \eta_i\right) - F^0$$

$$(4) \qquad d\xi_i/dt = \partial \bar{F}/\partial \eta_i, \quad d\eta_i/dt = -\partial \bar{F}/\partial \xi_i, \qquad i = 1, \cdots, n.$$

The equations of variation for this system have the form

$$(4') \qquad \begin{aligned} \frac{d\bar{\xi}_i}{dt} &= \sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{y}_i \partial \bar{x}_k} \bar{\xi}_k + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{y}_i \partial \bar{y}_k} \bar{\eta}_k \\ \frac{d\bar{\eta}_i}{dt} &= -\sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{x}_i \partial \bar{x}_k} \bar{\xi}_k - \sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{x}_i \partial \bar{y}_k} \bar{\eta}_k \end{aligned} \qquad i = 1, 2, \cdots, n$$

the coefficients being periodic functions of t, of the same period as the functions (2). If this period is T, and the functions (2) are not constants, it is well known that the characteristic exponents can be obtained by the construction of a fundamental set of solutions of (4'), $(\bar{\xi}_i^{(r)}, \bar{\eta}_i^{(r)})$, $(r = 1, \cdots, 2n)$ such that for $t = 0$,

$$\begin{aligned} &\bar{\xi}_s^{(r)} = 0, \quad r \neq s, \qquad \bar{\xi}_r^{(r)} = 1, \\ &\bar{\eta}_s^{(n+r)} = 0, \quad r \neq s, \quad \bar{\eta}_r^{(n+r)} = 1, \qquad r, s = 1, \cdots, n. \end{aligned}$$

Then if s is a root of the equation

$$= \left| \begin{array}{llllll} \bar{\xi}'_1(T)-s & \bar{\xi}''_1(T) & \cdots\ \bar{\xi}_1^{(n)}(T) & \bar{\xi}_1^{(n+1)}(T) & \cdots\ \bar{\xi}_1^{(2n)}(T) \\ \bar{\xi}'_2(T) & \bar{\xi}''_2(T)-s & \cdots\ \bar{\xi}_2^{(n)}(T) & \bar{\xi}_2^{(n+1)}(T) & \cdots\ \bar{\xi}_2^{(2n)}(T) \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ \bar{\xi}'_n(T) & \bar{\xi}''_n(T) & \cdots\ \bar{\xi}_n^{(n)}(T)-s & \bar{\xi}_n^{(n+1)}(T) & \cdots\ \bar{\xi}_n^{(2n)}(T) \\ \bar{\eta}'_1(T) & \bar{\eta}''_1(T) & \cdots\ \bar{\eta}_1^{(n)}(T) & \bar{\eta}_1^{(n+1)}(T)-s & \cdots\ \bar{\eta}_1^{(2n)}(T) \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ \bar{\eta}'_n(T) & \bar{\eta}''_n(T) & \cdots\ \bar{\eta}_n^{(n)}(T) & \bar{\eta}_n^{(n+1)}(T) & \cdots\ \bar{\eta}_n^{(2n)}(T)-s \end{array} \right.$$

the period T can be so chosen that every real root is positive.* The exponent corresponding to a root s is determined from the equation

$$s = e^{aT}$$

and these exponents occur in pairs † $(a, -a)$. The exponents corresponding to conjugate imaginary roots of $\Delta(s) = 0$ can evidently be grouped in conjugate pairs, if F is a real function of its arguments.

In the developments which follow it will be assumed that the elementary divisors of the matrix formed from the elements of the determinant above are linear; under this hypothesis, and the further assumption that no exponent is congruent to zero, (mod $2\pi i/T$) a fundamental set of solutions of (4′) can be represented in the form

$$(5) \qquad \begin{array}{ll} \bar{\xi}_i^{(r)} = e^{a_r t}\xi_i^{(r)}, \quad \bar{\eta}_i^{(r)} = e^{a_r t}\eta_i^{(r)}, & i = 1, \cdots, n, \\ \bar{\xi}_i^{(n+r)} = e^{-a_r t}\xi_i^{(n+r)}, \quad \bar{\eta}_i^{(n+r)} = e^{-a_r t}\eta_i^{(n+r)}, & r = 1, \cdots, n, \end{array}$$

in which the functions $\xi_i^{(s)}$, $\eta_i^{(s)}$ are periodic functions, of period T.

If the functions ϕ_i, ψ_i are constants, the exponents are defined as the roots of the equation

$$\left| \begin{array}{cccccc} \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{x}_1}-a, & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{x}_n} & \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{y}_1} & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{y}_n} \\ \cdot & \cdot\quad\cdot & \cdot & \cdot & \cdot\quad\cdot & \cdot \\ \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{x}_1} & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{x}_n}-a, & \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{y}_1} & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{y}_n} \\ -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{x}_1} & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{x}_n} & -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{y}_1}-a, & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{y}_n} \\ \cdot & \cdot\quad\cdot & \cdot & \cdot & \cdot\quad\cdot & \cdot \\ -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{x}_1} & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{x}_n} & -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{y}_1} & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{y}_n}-a \end{array} \right| = 0.$$

---

* Goursat-Hedrick, *Mathematical Analysis*, Vol. II, Part II, p. 149, also pp. 164-167.

† Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, tome 1, p. 196.

It will be assumed that the elementary divisors of the matrix corresponding to this determinant are linear, and that no root is zero. With these hypotheses the solutions of (4′) are in the form (5), the functions $(\xi_i^{(s)}, \eta_i^{(s)})$ being constants.

From the functions of (5) can be constructed a *normal* set possessing the following properties: if the real part of a_r is zero, the functions $\xi_i^{(r)}, \xi_i^{(n+r)}$ are conjugate imaginaries, as are also $\eta_i^{(r)}, \eta_i^{(n+r)}$; either the real part of a_r is positive, or $a_r/\sqrt{-1}$ is positive, $r=1, \cdots, n$; if a_r and a_s are conjugates, $\xi_i^{(r)}, \xi_i^{(s)}$ are conjugates, as are also $\eta_i^{(r)}, \eta_i^{(s)}$; and finally,

$$(6) \qquad \sum_{i=1}^{n} (\xi_i^{(r)}\eta_i^{(s)} - \eta_i^{(r)}\xi_i^{(s)}) = 0, \qquad s \neq n+r \text{ and } r \neq n+s$$
$$= \epsilon_r \neq 0, \quad s = n+r, \quad r = 1, \cdots, n$$

where $\epsilon_r = 1/a_r$ if a_r is real, $\epsilon_r = 1/2a_r$ if a_r is neither real nor pure imaginary, and $\epsilon_r = \pm 1/a_r$ if a_r is a pure imaginary.

For, as a result of the relation between any pair of solutions of (4′), we obtain

$$\sum_{i=1}^{n} (\bar{\xi}_i^{(r)}\bar{\eta}_i^{(s)} - \bar{\xi}_i^{(r)}\bar{\eta}_i^{(s)}) = C$$

$$\sum_{i=1}^{n} (\xi_i^{(r)}\eta_i^{(s)} - \eta_i^{(r)}\xi_i^{(s)}) = 0, \quad a_r + a_s \neq 0.$$

The exponents can evidently be chosen as required, and the functions $\xi_i^{(s)}$, $\eta_i^{(s)}$ can be grouped in the required manner without loss of generality. It remains to satisfy (6). From the investigations of Liapounoff and others it is known that if there are m sets corresponding to a_r, there are m corresponding to $-a_r$; if the corresponding sets of functions are $(\xi_i^{(r)}, \eta_i^{(r)})$, $(\xi_i^{(n+r)}, \eta_i^{(n+r)})$, $(r=1, \cdots, m)$, suppose first that the real part of a_r is not zero. The functions of (5) being linearly independent, some k must exist for which

$$\sum_{i=1}^{n} (\xi'_i\eta_i^{(n+k)} - \eta'_i\xi_i^{(n+k)}) \neq 0, \quad 1 \leq k \leq m$$

and for convenience suppose $k=1$. If functions

$$\xi_i = \sum_{r=1}^{m} C_r\xi_i^{(n+r)}, \quad \eta_i = \sum_{r=1}^{m} C_r\eta_i^{(n+r)}, \qquad i = 1, \cdots, n$$

are constructed, the constants $C_1, \cdots, C_m$ can be so chosen that

$$\sum_{i=1}^{n} (\xi'_i\eta_i - \eta'_i\xi_i) = \sum_{r=1}^{m} C_r \sum_{i=1}^{n} (\xi'_i\eta_i^{(n+r)} - \eta'_i\xi_i^{(n+r)}) = 0.$$

The coefficient of C_1 being different from zero, this equation has $(m-1)$

independent solutions, $(C_1{}^{(r)}, C_2{}^{(r)}, \cdots, C_m{}^{(r)})$ and the corresponding sets of functions $(\bar{\xi}_i{}^{(n+r)}, \bar{\eta}_i{}^{(n+r)})$ are independent sets.[*]

Similarly if[†]

$$\bar{\xi}_i = \sum_{r=1}^{m} C_r \xi_i{}^{(r)}, \quad \bar{\eta}_i = \sum_{r=1}^{m} C_r \eta_i{}^{(r)} \quad i = 1, \cdots, n$$

the equation

$$\sum_{i=1}^{n} (\xi_i{}^{(n+1)} \bar{\eta}_i - \eta_i{}^{(n+1)} \bar{\xi}_i) = \sum_{r=1}^{m} C_r \sum_{i=1}^{n} (\xi_i{}^{(n+1)} \eta_i{}^{(r)} - \eta_i{}^{(n+1)} \xi_i{}^{(r)}) = 0$$

has $(m-1)$ independent solutions, and the sets of functions $(\bar{\xi}_i{}^{(s)}, \bar{\eta}_i{}^{(s)})$, $(s = 2, \cdots, m)$ are linearly independent. With the $2(m-1)$ sets thus obtained one may proceed in the same manner, until a set has been obtained for which

$$\begin{aligned} \sum_{i=1}^{n} (\xi_i{}^{(r)} \eta_i{}^{(s)} - \eta_i{}^{(r)} \xi_i{}^{(s)}) &= 0, \quad s \neq n + r, \\ &= a_r \neq 0, \quad s = n + r, \quad r = 1, \cdots, m; \end{aligned}$$

if the functions of each set $(\xi_i{}^{(r)}, \eta_i{}^{(r)})$ are multiplied by a suitable constant, the equations

$$\begin{aligned} \sum_{i=1}^{n} (\xi_i{}^{(r)} \eta_i{}^{(n+r)} - \eta_i{}^{(r)} \xi_i{}^{(n+r)}) &= 1/a_r, \ a_r \text{ real}, \\ &= 1/2a_r, \ R(a_r/i) \neq 0, \ r = 1, \cdots, m \end{aligned}$$

are obtained.

If a_r is a pure imaginary, the corresponding functions of the sets $(\xi_i{}^{(r)}, \eta_i{}^{(r)})$, $(\xi_i{}^{(n+r)}, \eta_i{}^{(n+r)})$ may be assumed conjugates; if

$$\sum_{i=1}^{n} (\xi_i{}^{(r)} \eta_i{}^{(n+r)} - \eta_i{}^{(r)} \xi_i{}^{(n+r)}) = 0, \quad r = 1, \cdots, m,$$

then

$$\sum_{i=1}^{n} (\xi'_i \eta_i{}^{(n+s)} - \eta'_i \xi_i{}^{(n+s)}) = a\sqrt{-1}, \ a \neq 0, \ 1 < s \leq m.$$

and taking conjugates, a being real,

$$\sum_{i=1}^{n} (\xi_i{}^{(n+1)} \eta_i{}^{(s)} - \eta_i{}^{(n+1)} \xi_i{}^{(s)}) = -a\sqrt{-1}.$$

Consequently if

$$\begin{aligned} &\bar{\xi}'_i = \xi'_i + \xi_i{}^{(s)}, \quad \bar{\eta}'_i = \eta'_i + \eta_i{}^{(s)}, \\ &\qquad \bar{\xi}_i{}^{(n+1)} = \xi_i{}^{(n+1)} + \xi_i{}^{(n+s)}, \quad \bar{\eta}_i{}^{(n+1)} = \eta_i{}^{(n+1)} + \eta_i{}^{(n+s)}, \end{aligned}$$

then

$$\sum_{i=1}^{n} (\bar{\xi}'_i \bar{\eta}_i{}^{(n+1)} - \bar{\eta}'_i \bar{\xi}_i{}^{(n+1)}) = 2a\sqrt{-1};$$

---

[*] Bocher, *Introduction to Higher Algebra*, (1922), p. 50.

[†] The dashed variables used here are not to be confused with those of (5).

and if each function of the sets $(\bar\xi'_i, \bar\eta'_i)$; $\bar\xi_i^{(n+1)}, \bar\eta^{(n+1)})$ is multiplied by the same real constant, suitably chosen, the relation

$$\sum_{i=1}^{n} (\bar\xi'_i\bar\eta_i^{(n+1)} - \bar\eta'_i\bar\xi_i^{(n+1)}) = \pm 1/a_1$$

is obtained. Hence this relation must hold for at least one pair of conjugate sets. Then as before, from the equation

$$\sum_{r=1}^{m} C_r \sum_{i=1}^{n} (\bar\xi'_i\eta_i^{(n+r)} - \bar\eta'_i\xi_i^{(n+r)}) = 0$$

and its conjugate

$$\sum_{r=1}^{m} \bar C_r \sum_{i=1}^{n} (\bar\xi_i^{(n+1)}\eta_i^{(r)} - \bar\eta_i^{(n+1)}\xi_i^{(r)}) = 0$$

can be determined linearly independent sets of functions $(\bar\xi_i^{(r)}, \bar\eta_i^{(r)})$, $(\bar\xi_i^{(n+r)}, \bar\eta_i^{(n+r)})$, $(r = 2, \cdots, m)$ which can be operated on as before until sets have been obtained for which

$$\begin{aligned} \sum_{i=1}^{n} (\bar\xi_i^{(r)}\bar\eta_i^{(n+s)} - \bar\eta_i^{(r)}\bar\xi_i^{(n+s)}) &= 0, \ s \neq r, \quad r, s = 1, \cdots, m, \\ &= \pm 1/a_r, \quad s = r, \end{aligned}$$

the functions $\bar\xi_i^{(r)}, \bar\xi_i^{(n+r)}$ being conjugates, and the same is true of $\bar\eta_i^{(r)}, \bar\eta_i^{(n+r)}$.

When one or the other procedure has been applied to each exponent, conjugate functions $\xi_i^{(r)}$, $\xi_i^{(s)}$, $\eta_i^{(r)}$, $\eta_i^{(s)}$ being chosen to correspond to conjugate imaginary exponents as required, the resulting set will be a normal one; and in what follows it will be assumed that the periodic functions or constants in (5) form a normal set.

2. Reduction of the differential equations to a simple form.

Equations (4), (4′) can now be reduced to a simpler form by means of the change of variables

$$(7) \qquad \begin{aligned} \xi_i &= \sum_{r=1}^{n} \xi_i^{(r)}\bar u_r + \sum_{r=1}^{n} \xi_i^{(n+r)}\bar v_r \\ \eta_i &= \sum_{r=1}^{n} \eta_i^{(r)}\bar u_r + \sum_{r=1}^{n} \eta_i^{(n+r)}\bar v_r \end{aligned} \qquad i = 1, \cdots, n,$$

after which (4) become

$$(8) \qquad d\bar u_r/dt = a_r\bar u_r + \phi'_r, \quad d\bar v_r/dt = -a_r\bar v_r + \psi'_r \qquad r = 1, \cdots, n.$$

If a is an arbitrary constant of integration, from (6), (7), we obtain

$$\frac{\partial \bar{F}}{\partial a} = \sum_{i=1}^{n} \left(\frac{d\xi_i}{dt} \frac{\partial \eta_i}{\partial a} - \frac{d\eta_i}{dt} \frac{\partial \xi_i}{\partial a} \right)$$

$$= \frac{\partial \Omega}{\partial a} + \sum_{i=1}^{n} \epsilon_r \left(\frac{d\bar{u}_r}{dt} \frac{\partial \bar{v}_r}{\partial a} - \frac{d\bar{v}_r}{dt} \frac{\partial \bar{u}_r}{\partial a} \right)$$

where

$$\Omega = \sum_{i=1}^{n} \frac{\bar{u}_r^2}{2} \sum_{i=1}^{n} \left(\frac{d\xi_i^{(r)}}{dt} \eta_i^{(r)} - \frac{d\eta_i^{(r)}}{dt} \xi_i^{(r)} \right)$$
$$+ \sum_{r<s} \bar{u}_r \bar{u}_s \sum_{i=1}^{n} \left(\frac{d\xi_i^{(r)}}{dt} \eta_i^{(s)} - \frac{d\eta_i^{(r)}}{dt} \xi_i^{(s)} \right)$$
$$+ \sum_{r=1}^{n} \frac{\bar{v}_r^2}{2} \sum_{i=1}^{n} \left(\frac{d\xi_i^{(n+r)}}{dt} \eta_i^{(n+r)} - \frac{d\eta_i^{(n+r)}}{dt} \xi_i^{(n+r)} \right)$$
$$+ \sum_{r<s} \bar{v}_r \bar{v}_s \sum_{i=1}^{n} \left(\frac{d\xi_i^{(n+r)}}{dt} \eta_i^{(n+s)} - \frac{d\eta_i^{(n+r)}}{dt} \xi_i^{(n+s)} \right)$$
$$+ \sum_{r,s=1}^{n} \bar{u}_r \bar{v}_s \sum_{i=1}^{n} \left(\frac{d\xi_i^{(r)}}{dt} \eta_i^{(n+s)} - \frac{d\eta_i^{(r)}}{dt} \xi_i^{(n+s)} \right)$$

Consequently * if $F' = \bar{F} - \Omega$, the equations in $\bar{u}_i$, $\bar{v}_i$, are in the form

$$(9) \qquad \frac{d\bar{u}_i}{dt} = \frac{\partial F'}{\partial (\epsilon_i \bar{v}_i)}, \quad \epsilon_i \frac{d\bar{v}_i}{dt} = -\frac{\partial F'}{\partial \bar{u}_i} \qquad i = 1, \cdots, n$$

and from (8),

$$(10) \qquad F' = \sum_{r=1}^{n} \epsilon_r a_r \bar{u}_r \bar{v}_r + F_3$$

If the functions $(\xi_i^{(s)}, \eta_i^{(s)})$ are constants, $\Omega = 0$; in any case

$$(11) \qquad \frac{\partial^2 F_3}{\partial \bar{u}_r \partial \bar{u}_s} = \frac{\partial^2 F_3}{\partial \bar{u}_r \partial \bar{v}_s} = \frac{\partial^2 F_3}{\partial \bar{v}_r \partial \bar{v}_s} = 0, \quad \bar{u}_r = \bar{u}_s = 0, \quad r, s, = 1, \cdots, n.$$

The function F having been defined only for real values of ξ_i, η_i, it is necessary to give $\bar{u}_i$, $\bar{v}_i$ values for which the original variables are real. This requirement will be satisfied if, when a_r and a_s are conjugate imaginaries, corresponding functions of the pairs $(\bar{u}_r, \bar{v}_r)$, $(\bar{u}_s, \bar{v}_s)$ are given conjugate values; while if a_r is a pure imaginary, the variables $\bar{u}_r$, $\bar{v}_r$ are given conjugate values, as a result of the hypothesis that the functions of (5) are a normal set.

---

* Poincaré, *Leçons de Mécanique Céleste*, tome 1, pp. 2, 3; *Les Méthodes Nouvelles*, tome 1, pp. 368, 369.

If a_r, a_s are conjugates, suppose

$$\begin{aligned} a_r &= q + iq' & \bar{u}_r &= u' + iu'' & \bar{v}_r &= v' + iv'' \\ a_s &= q - iq' & \bar{u}_s &= u' - iu'' & \bar{v}_s &= v' - iv'' \end{aligned}$$

then from (8)

$$\begin{aligned} du'/dt &= qu' - q'u'' + \phi' & dv'/dt &= -qv' + q'v'' + \psi' \\ du''/dt &= q'u' + qu'' + \phi'' & dv''/dt &= -q'v' - qv'' + \psi'' \end{aligned}$$

If a_r is real, and $a_r = q_r$,

$$d\bar{u}_r/dt = q_r\bar{u}_r + \bar{\bar{\phi}}'_r \qquad d\bar{v}_r/dt = -q_r\bar{v}_r + \bar{\bar{\psi}}'_r$$

while if a_r is a pure imaginary, $a_r = p\sqrt{-1}$, $p > 0$,

$$\bar{u} = x + iy, \quad \bar{v} = x - iy,$$
$$dx/dt = -py + \bar{\phi}', \quad dy/dt = px + \bar{\psi}'.$$

Of the exponents $a_1, a_2, \cdots, a_n$ assume that there are M pure imaginaries, K real, and L pairs of conjugates; then $K + 2L = N$, and $M + N = n$. Equations (8) can then be written in the form

$$\begin{aligned} &dx_i/dt = -p_iy_i + X_i, & &dy_i/dt = p_ix_i + Y_i & &i = 1, \cdots, M, \\ &du_k/dt = q_ku_k + U_k, & &dv_k/dt = -q_kv_k + V_k, & &k = 1, \cdots, K, \\ (12)\quad &du_r/dt = q_ru_r - q'_ru_{L+r} + U_r, & &dv_r/dt = -q_rv_r + q'_rv_{L+r} + V_r & & \\ &du_{L+r}/dt = q'_ru_r + q_ru_{L+r} + U_{L+r}, & & & &(r = K+1, \cdots, K+L) \\ & & &dv_{L+r}/dt = -q'_rv_r - q_rv_{L+r} + V_{L+r}\,. & & \end{aligned}$$

From (8), (9), (10), it is seen that F' has the form

$$(13)\qquad F' = \sum_{i=1}^{M} \pm (x^2_i + y^2_i) + \sum_{m=1}^{K} u_mv_m + \sum_{r=K}^{K+L} (u_rv_r - u_{L+r}v_{L+r}) + F_3$$

Since the absolute values of the coefficients of (7) are bounded, a number a can be found such that if

$$(D')\qquad |x_i| < a, \quad |y_i| < a, \quad |u_r| < a, \quad |v_r| < a,$$
$$i = 1, \cdots, M, \ r = 1, \cdots, N,$$

the corresponding variables ξ_i, η_i will satisfy the conditions $|\xi_i| < d'$, $|\eta_i| < d'$.

Since the functions

$$(14)\qquad X_i, \ Y_i, \ U_r, \ V_r, \qquad i = 1, \cdots, M, \quad r = 1, \cdots, N$$

are real, and from (8, 9, 10) are necessarily linear combinations of the first order partial derivatives of F_3, they possess continuous first order partial derivatives with respect to the variables now considered, which vanish with these variables. If we denote by $g(x_i, y_i, u_r, v_r)$ the largest of the absolute values of the partial derivatives of the functions (14), at a point (x_i, y_i, u_r, v_r) of (D'), then for some τ, $0 \leq \tau \leq 1$,

$$(15) \qquad |X_i| \leq g\,(\tau x_i, \tau y_i, \tau u_r, \tau v_r) \left[\sum_{i=1}^{M} (|x_i| + |y_i|) + \sum_{r=1}^{N} (|u_r| + |v_r|) \right.$$

and similar inequalities hold for the other functions of (14).

Instead of a single trajectory consider two, represented by the variables

$$x_i,\ y_i,\ u_r,\ v_r; \quad x_i + x'_i,\ y_i + y'_i, \quad u_r + u'_r,\ v_r + v'_r;$$

from (12) we obtain

$$(16) \qquad \begin{aligned} &dx'_i/dt = -p_i y'_i + X'_i, \qquad dy'_i/dt = p_i x'_i + Y'_i, \quad i = 1, \cdots, M, \\ &du'_m/dt = q_m u'_m + U'_m, \quad dv'_m/dt = -q_m v'_m + V'_m, \quad m = 1, \cdots, K \\ &du'_r/dt = q_r u'_r - q'_r u'_{L+r} + U'_r, \quad dv'_r/dt = -q_r v'_r + q'_r v'_{L+r} + V'_r, \\ &du'_{L+r}/dt = q'_r u'_r + q_r u'_{L+r} + U'_{L+r}, \\ &\qquad\qquad dv'_{L+r}/dt = -q'_r v'_r - q_r v'_{L+r} + V'_{L+r}, \\ &\qquad\qquad\qquad (r = K+1, \cdots, K+L). \end{aligned}$$

the primed functions denoting the differences between the corresponding functions (14) for the two points considered. From the equations above we obtain immediately,

$$(17) \qquad \begin{aligned} &\frac{d}{dt}(x'^2_i + y'^2_i) = 2(x'_i X'_i + y'_i Y'_i) \qquad i = 1, \cdots, n. \\ &\frac{du'^2_m}{dt} = 2q_m u'^2_m + 2u'_m U'_m, \\ &\qquad \frac{dv'^2_m}{dt} = -2q_m v'^2_m + 2v'_m V'_m \qquad m = 1, \cdots, K. \\ &\frac{d}{dt}(u'^2_r + u'^2_{L+r}) = 2q_r(u'^2_r + u'^2_{L+r}) + 2(u'_r U'_r + u'_{L+r} U'_{L+r}), \\ &\frac{d}{dt}(v'^2_r + v'^2_{L+r}) = -2q_r(v'^2_r + v'^2_{L+r}) + 2(v'_r V'_r + v'_{L+r} V'_{L+r}) \\ &\qquad\qquad (r = K+1, \cdots, K+L). \end{aligned}$$

As before,

$$(18) \qquad \begin{aligned} &|X'_i| \leq g\,(x_i + \tau x'_i, y_i + \tau y'_i, u_r + \tau u'_r, v_r + \tau v'_r) \\ &\qquad \left[\sum_{i=1}^{M} (|x'_i| + |y'_i|) + \sum_{r=1}^{N} (|u'_r| + |v'_r|) \right] \qquad 0 < \tau < 1. \end{aligned}$$

with similar inequalities for the other functions.

Part II. *Extension of the theorems of Bohl.*

1. Some preliminary inequalities.

The paragraphs which follow are very similar to the corresponding ones in the memoir of Bohl.* Suppose $l_1, l_2, \cdots, l_M, m_1, m_2, \cdots, m_N, m_{K+L+r} = m_{K+r}$ any positive numbers, and $s_1, s_2, \cdots, s_M, t_1, t_2, \cdots, t_M, z_1, z_2, \cdots, z_N$ a set of real variables. Then from the inequality

$$\sqrt{m}\sqrt{\sum_{i=1}^{m} x^2_i} \geq \sum_{i=1}^{m} | x_i |$$

and the introduction of the constants

$$\bar{k} \leq l_i, m_r, \quad \bar{g} \geq l_i, m_r, \quad \Gamma = \frac{1}{\bar{k}}\sqrt{2M+N}, \quad \Delta = \bar{g}\sqrt{2M+N}$$

we obtain

$$(19)\quad \begin{aligned} \Gamma\sqrt{\sum_{i=1}^{M} l^2_i(s^2_i + t^2_i) + \sum_{r=1}^{N} m^2_r z^2_r} &\geq \sum_{i=1}^{M} (| s_i | + | t_i |) + \sum_{r=1}^{N} | z_r | \\ \Delta\sqrt{\sum_{i=1}^{M} l^2_i(s^2_i + t^2_i) + \sum_{r=1}^{N} m^2_r z^2_r} &\geq \sum_{i=1}^{M} l^2_i(| s_i | + | t_i |) + \sum_{r=1}^{N} m^2_r | z_r | . \end{aligned}$$

By choosing the constant a defining (D') sufficiently small we can make the quantity $g(x_i, y_i, u_r, v_r)$ arbitrarily small; hence if λ is a positive quantity to be restricted later, we may assume $g < \lambda$. For convenience suppose $q_{K+r} = q_{K+L+r}$. If

$$r' = \sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r - \sum_{r=1}^{N} u'^2_r$$

we obtain from (17),

$$\begin{aligned} \frac{1}{2}\cdot\frac{dr'}{dt} = &-\sum_{r=1}^{N} m^2_r q_r v'^2_r - \sum_{r=1}^{N} q_r u'^2_r \\ &+ \sum_{i=1}^{M} l^2_i(x'_i X'_i + y'_i Y'_i) + \sum_{r=1}^{N} m^2_r v'_r V'_r - \sum_{r=1}^{N} u'_r U'_r . \end{aligned}$$

Since

$$| X'_i |, \ | Y' |, \ | U'_r |, \ | V'_r | \leq \lambda\left[\sum_{i=1}^{M}(| x'_i | + | y'_i |) + \sum_{r=1}^{N}(| u'_r | + | v'_r |)\right]$$

and

$$\sum_{i=1}^{M}(| x'_i | + | y'_i |) + \sum_{r=1}^{N} | v'_r | \leq \Gamma\sqrt{\sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r}$$

$$\sum_{r=1}^{N} | u'_r | \leq \sqrt{N}\sqrt{\sum_{r=1}^{N} u'^2_r}$$

---

* *L. c.*, II, § 6.

we obtain

$$|X'_i|,\ |Y'_i|,\ |U'_r|,\ |V'_r| \leq \lambda \left[\Gamma\sqrt{\sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r} + \sqrt{N}\sqrt{\sum_{r=1}^{N} u'^2_r}\right]$$

Consequently if $r' \leq 0$ when $t = t_0$,

$$\frac{1}{2}\frac{dr'}{dt} \leq -\sum_{r=1}^{N} q_r u'^2_r + \lambda(\Gamma + \sqrt{N})\sqrt{\sum_{r=1}^{N} u'^2_r} \left[\sum_{i=1}^{M} l^2_i(|x'_i| + |y'_i|) + \sum_{r=1}^{N}(m^2_r|v'_r| + |u'_r|)\right]$$

Since

$$\sum_{i=1}^{M} l^2_i(|x'_i| + |y'_i|) + \sum_{r=1}^{N} m^2_r|v'_r| \leq \Delta\sqrt{\sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r} \leq \Delta\sqrt{\sum_{r=1}^{N} u'^2_r}$$

we have

$$\frac{1}{2}\frac{dr'}{dt} \leq -\sum_{r=1}^{N} q_r u'^2_r + \lambda(\Gamma + \sqrt{N})(\Delta + \sqrt{N})\sum_{r=1}^{N} u'^2_r.$$

Then if q is the smallest of the quantities $q_1, q_2, \cdots, q_N$,

$$\frac{1}{2}\frac{dr'}{dt} \leq -[q - \lambda(\Gamma + \sqrt{N})(\Delta + \sqrt{N})]\sum_{r=1}^{N} u'^2_r.$$

or if $\lambda(\Gamma + \sqrt{N})(\Delta + \sqrt{N}) < q$, then $dr'/dt \leq 0$.

Suppose this inequality for λ satisfied within (D'). Then if the two trajectories lie in (D') for $t_0 \leq t \leq t_1$, and $r' \leq 0$ for $t = t_0$, then $dr'/dt < 0$ also if $\sum_{i=1}^{N} u'^2_r \neq 0$. Suppose $r' \geq 0$ for $t = t_1$; then a time $t' \leq t_1$ must exist such that $r' = 0$, $t = t'$, and $r' < 0$, $t_0 < t < t'$. Consequently there exists also a time $t'' < t'$ at which $dr'/dt = 0$; but at this time $r' < 0$, hence $dr'/dt < 0$. From this contradiction we obtain the result that $r' < 0$ when $t = t_1$, and $r' < 0$, $t_0 < t \leq t_1$.

Consequently from the definition of r', $\sum_{r=1}^{N} u'^2_r > 0$, $t_0 < t \leq t_1$, and if $\sum_{r=1}^{N} u'^2_r \neq 0$ for $t = t_0$, then $\sum_{r=1}^{N} u'^2_r > 0$, $t_0 \leq t \leq t_1$.

Under these assumptions compute $d/dt \sum_{r=1}^{N} u'^2_r$:

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u'^2_r = \sum_{r=1}^{K} (q_r u'^2_r + u'_r U'_r)$$

$$+ \sum_{r=K+1}^{K+L} [q_r(u'^2_r + u'^2_{L+r}) + u'_r U'_r + u'_{L+r} U'_{L+r}]$$

$$= \sum_{r=1}^{N} q_r u'^2_r + \sum_{r=1}^{N} u'_r U'_{ri}$$

Since

$$\left|\sum_{r=1}^{N} u'_r U'_r\right| \leq \left(\sum_{r=1}^{N} |u'_r|\right) \lambda \left[\sum_{i=1}^{M} (|x'_i| + |y'_i|) + \sum_{r=1}^{N} (|u'_r| + |v'_r|)\right]$$

$$\leq \lambda\sqrt{N}\sqrt{\sum_{r=1}^{N} u'^2_r}\,(\Gamma + \sqrt{N})\sqrt{\sum_{r=1}^{N} u'^2_r}$$

$$\leq \lambda\sqrt{N}\,(\Gamma + \sqrt{N}) \sum_{r=1}^{N} u'^2_r$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u'^2_r \geq [q - \lambda\sqrt{N}(\Gamma + \sqrt{N})] \sum_{r=1}^{N} u'^2_r$$

Suppose $\sigma > 2\lambda\sqrt{N}(\Gamma + \sqrt{N})$, while $\sigma < 2q$; then

$$\frac{d}{dt}\sum_{r=1}^{N} u'^2_r > (2q - \sigma) \sum_{r=1}^{N} u'^2_r$$

and

$$\left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_1} > e^{(2q-\sigma)(t_1-t_0)} \left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_0}.$$

Consequently we have the following theorem: Given a set of positive numbers $\sigma, l_1, l_2, \cdots, l_M, m_1, m_2, \cdots, m_N$, with $m_{K+L+r} = m_{K+r}$, then if a neighborhood (D') of the origin is so chosen that within it the partial derivatives of X_i, Y_i, U_r, V_r are less in absolute value than either of the quantities

$$\frac{q}{(\Gamma + \sqrt{N})(\Delta + \sqrt{N})}, \quad \frac{\sigma}{2\sqrt{N}(\Gamma + \sqrt{N})}$$

where Γ, Δ are the quantities previously defined, this neighborhood has the following properties: 1. If the coördinates of two trajectories lie in (D') for $t_0 \leq t \leq t_1$ and if when $t = t_0$

$$r' = \sum_{i=1}^{M} l^2_i (x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r - \sum_{r=1}^{N} u'^2_r \leq 0, \quad \sum_{r=1}^{N} u'^2_r \neq 0,$$

then $r' < 0$ for the interval $t_0 < t \leq t_1$; and

$$\left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_1} > e^{(2q-\sigma)(t_1-t_0)} \left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_0}$$

2. If the coördinates defining two trajectories lie in (D') for $t_0 \geq t \geq t_1$ $(t_1 < t_0)$ and if for $t = t_0$,

$$s' = \sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r u'^2_r - \sum_{r=1}^{N} v'^2_r \leq 0, \; \sum_{r=1}^{N} v'^2_r \neq 0,$$

then $s' < 0$ for the interval $t_0 > t \geq t_1$, and

$$\left[\sum_{r=1}^{N} v'^2_r\right]_{t=t_1} > e^{(2q-\sigma)(t_0-t_1)} \left[\sum_{r=1}^{N} v'^2_r\right]_{t=t_0}$$

The last part of this theorem can be demonstrated in the same manner as the first.*

2. Certain inequalities obtained from the energy integral.

In this and the following paragraphs it will be assumed that the functions ϕ_i, ψ_i in (2) are constants, and that the function F does not contain the time, consequently the integral $F = C$ exists; it will also be assumed that the quadratic form $\sum_{i=1}^{M} \pm (x^2_i + y^2_i)$ in (13) is definite. Then if $\mu > 0$ is given, the constant a defining (D') can be chosen so small that within this region, (E),

$$|F_3| \leq \mu \left[\sum_{i=1}^{M} (x^2_i + y^2_i) + \sum_{r=1}^{N} (u^2_r + v^2_r)\right].$$

Suppose δ a positive number such that the region defined by the inequalities

$$(F_1) \qquad \sum_{i=1}^{M} (x^2_i + y^2_i) \leq \delta^2$$

$$(G) \qquad \sum_{r=1}^{N} u^2_r \leq \delta^2/4, \quad \sum_{r=1}^{N} v^2_r \leq \delta^2/4$$

---

* While it is almost self-evident that no coördinate can become indeterminate for a value $\bar{t}$ on the interval $t_0 < t < t_1$ under these hypotheses, a proof can be given as follows. Suppose M' an upper bound for the right-hand members of (12), and $|x_i|$, $|y_i|$, $|u_r|$, $|v_{v_r}| < b$ in the neighborhood considered. Then if t' is any value of t, the method of successive approximations of Picard defines these coördinates within an interval $|t'' - t'| < b/M'$. Since any finite value of $t - t_0$ is less than some integral multiple of b/M', the values of the coördinates for this time $\bar{t}$ are defined by the method of Picard, and cannot therefore be indeterminate when $t \to \bar{t}$, if it is assumed that the cordinates lie in the given region for $t_0 \leq t \leq \bar{t} + \bar{h}$, $\bar{h} > 0$.

lies within (E). Introduce also the region

$$(F_2) \qquad \sum_{r=1}^{M} (x^2_i + y^2_i) \leq 2\delta^2/9.$$

Then from the energy integral and (13) we obtain the theorem: If the trajectory (x_i, y_i, u_r, v_r) lies in the region (F_1, G) for $t = t_1$, and in (F_2, G) for $t = t_2$, then if $\mu < \frac{1}{8}$,

$$\left[\sum_{i=1}^{M} (x^2_i + y^2_i)\right]_{t=t_1} < \delta^2.$$

For from (13),

$$\begin{aligned}\left[\sum_{i=1}^{M} (x^2_i + y^2_i)\right]_{t=t_1} &= \pm \left[\pm \sum_{i=1}^{M} (x^2_i + y^2_i) + \sum_{m=1}^{K} u_m v_m + \sum_{r=K+1}^{K+L} (u_r v_r - u_{L+r} v_{L+r}) + F_3\right]_{t=t_2} \\ &\qquad \mp \left[\sum_{m=1}^{K} u_m v_m + \sum_{r=K+1}^{K+L} (u_r v_r - u_{L+r} v_{L+r}) + F_3\right]_{t=t_1}\end{aligned}$$

Hence from the inequalities $u_k v_k \leqq \frac{1}{2}(u^2_k + v^2_k)$, $-u_r v_r \leq \frac{1}{2}(u^2_r + v^2_r)$, we obtain

$$\begin{aligned}\left[\sum_{i=1}^{M} (x^2_i + y^2_i)\right]_{t=t_1} &\leq \frac{2\delta^2}{9} + \frac{\delta^2}{2} + \mu \left[2\delta^2 + \frac{2\delta^2}{9}\right] \\ &\leq \delta^2[2/9 + 1/2 + \mu(2 + 2/9)]\end{aligned}$$

and if $\mu < \frac{1}{8}$, the theorem follows.

3. Introduction of certain domains and inequalities.

Suppose (H) a region lying within (E) and such that in (H) the partial derivatives of the functions (14) are less in absolute value than θq, where

$$\theta < \frac{1}{2\sqrt{N}(\sqrt{2M} + \sqrt{N})}$$

Assume d so small that the region

$$\sum_{i=1}^{M} (x^2_i + y^2_i) \leq d^2, \quad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4$$

lies within (H), and construct the regions

$$(A) \qquad \sum_{i=1}^{M} (x^2_i + y^2_i) \leq d^2,$$

$$(a) \qquad \sum_{i=1}^{M} (x^2_i + y^2_i) \leq 2d^2/9,$$

$$(B) \qquad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4.$$

Then we have the theorem: If (x_i, y_i, u_r, v_r) lies in (a, B) for $t = t_2$, in (A, B) for $t = t_1$, and if for $t = t_1$, $\sum_{r=1}^{N} u^2_r = d^2/4$, then $d/dt \sum_{r=1}^{N} u^2_r > 0$.

For, from the preceding paragraph, if $t = t_1$

$$\sum_{i=1}^{M} (x^2_i + y^2_i) < d^2.$$

From (12) or (16) we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{r=1}^{N} u^2_r = \sum_{r=1}^{K} (q_r u^2_r + u_r U_r) + \sum_{r=K+1}^{K+L} [q_r (u^2_r + u^2_{L+r}) + u_r U_r + u_{L+r} U_{L+r}]$$

$$= \sum_{r=1}^{N} q_r u^2_r + \sum_{r=1}^{N} u_r U_r.$$

From the inequalities

$$\left| \sum_{r=1}^{N} u_r U_r \right| \leq \left(\sum_{r=1}^{N} | u_r | \right) \theta q \left[\sum_{i=1}^{M} (| x_i | + | y_i |) + \sum_{r=1}^{N} (| u_r | + | v_r |) \right]$$

$$\sum_{i=1}^{M} (| x_i | + | y_i |) \leq \sqrt{2M} \sqrt{\sum_{i=1}^{M} (x^2_i + y^2_i)}$$

$$\sum_{r=1}^{N} (| u_r | + | v_r |) \leq \sqrt{2N} \sqrt{\sum_{r=1}^{N} u^2_r + v^2_r)}$$

$$\sum_{r=1}^{N} | u_r | \leq \sqrt{N} \sqrt{\sum_{r=1}^{N} u^2_r}$$

we obtain

$$\left| \sum_{r=1}^{N} u_r U_r \right| \leq \theta q \sqrt{N} \sqrt{\sum_{r=1}^{N} u^2_r} \left[\sqrt{2M} \sqrt{\sum_{i=1}^{M} (x^2_i + y^2_i)} \right.$$

$$\left. + \sqrt{2N} \sqrt{\sum_{r=1}^{N} (u^2_r + v^2_r)} \right]$$

$$\leq \theta q \sqrt{N} \frac{d}{2} [\sqrt{2M}\, d + \sqrt{2N} \sqrt{d^2/2}]$$

$$\leq \theta q d^2 \frac{\sqrt{N}}{2} [\sqrt{2M} + \sqrt{N}] < q \frac{d^2}{4}$$

Also,

$$\sum_{r=1}^{N} q_r u^2_r \geq q \sum_{r=1}^{N} u^2 = q \frac{d^2}{4}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u^2{}_r > q\frac{d^2}{4} - q\frac{d^2}{4} = 0.$$

In the same manner can be proved the theorem: If (x_i, y_i, u_r, v_r) lies in (a, B) for $t = t_1$, in (A, B) for $t = t_2$, then if for $t = t_1$, $\sum_{r=1}^{N} v^2{}_r = d^2/4$,

$$\frac{d}{dt}\sum_{r=1}^{N} v^2{}_r < 0, \quad t = t_1.$$

4. Existence theorems.

For convenience the domains introduced above can be grouped in the form

$$(A, B) \qquad \sum_{i=1}^{M}(x^2{}_i + y^2{}_i) \leq d^2, \quad \sum_{r=1}^{N} u^2{}_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2{}_r \leq d^2/4,$$

$$(a, B) \qquad \sum_{i=1}^{M}(x^2{}_i + y^2{}_i) \leq 2d^2/9; \quad \sum_{r=1}^{N} u^2{}_r \leq d^2/4, \quad \sum_{r=1}^{N} v_r{}^2 \leq d^2/4.$$

We shall prove the following theorem: To every set of values of the region (R),

$$\sum_{i=1}^{M}(x^2{}_i + y^2{}_i) \leq 2d^2/9, \quad \sum_{r=1}^{N} v^2{}_r \leq d^2/4$$

can be adjoined a set of u-values of the region $\sum_{r=1}^{N} u^2{}_r \leq d^2/4$ such that the set of values (x_i, y_i, v_r), (u_r) are the initial values of a trajectory which remains in the region (A, B) for $t > t_0$.

The theorem will be true if it can be shown that the contrary assumption leads to a contradiction. Suppose $(x_i, y_i, v_r)_0$ a set of values of (R), such that however a set of values $(u)_0$ is chosen satisfying $\sum_{r=1}^{N} u^2{}_r \leq d^2/4$, the trajectory possessing the initial values $(x_i, y_i, v_r)_0$, $(u)_0$ ceases to satisfy the inequalities of (A, B) for $t > \tau$; in other words, inequalities (A, B) are satisfied for $t_0 \leq t \leq \tau$, while however small h may be, these inequalities are not satisfied in the entire interval $t_0 \leq t < \tau + h$. Consider the values of $u_1, u_2, \cdots, u_N$ when $t = \tau$; if this set is denoted by (u), it will be shown that (u) lies on the boundary of the region $\sum_{r=1}^{N} u^2{}_r \leq d^2/4$.

The set of values (x_i, y_i, u_r, v_r) lies on the boundary of (A, B) for $t = \tau$ and since the trajectory lies in (a, B) for $t = t_0$, we have from the results of a preceding paragraph $\sum_{i=1}^{M}(x^2{}_i + y^2{}_i) < d^2$. Also, if the equation

$\sum_{r=1}^{N} v^2_r = d^2/4$ is satisfied, then $d/dt \sum_{r=1}^{N} v^2_r < 0$; and if the set of values (u) did not lie on the boundary, $(\sum_{r=1}^{N} u^2_r < d^2/4)$, it would follow that the trajectory would remain in the region (A, B) for some interval of time, contrary to hypothesis. Hence to each set $(u)_0$ corresponds a set (u) on the boundary of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$. It will be shown that this correspondence is a continuous one.*

Suppose first that the set $(u)_0$ lies in the interior of the region, and consequently $\sum_{r=1}^{N} u^2_r < d^2/4$; then $\tau > t_0$. Then a constant h can be determined such that the trajectory with initial values $(x_i, y_i, v_r)_0(u)_0$ lies within (H) for $t_0 \leq t \leq \tau + h$, $\sum_{r=1}^{N} u_r^2 > d^2/4$ for $t = \tau + h$, and $t_0 < \tau - h$. The number h can be chosen arbitrarily small. In the interval $t_0 \leq t \leq \tau - h$ the inequality $\sum_{r=1}^{N} u^2_r < d^2/4$ is constantly satisfied, otherwise the trajectory would not satisfy $\sum_{r=1}^{N} u^2_r \leq d^2/4$ for $t_0 \leq t \leq \tau$, from the inequality $d/dt \sum_{r=1}^{N} u^2_r > 0$ on the boundary of this region.

In the interval $t_0 \leq t \leq \tau - h$, then, $d^2/4 - \sum_{r=1}^{N} u^2_r \geq \beta^2 > 0$; consequently a region (ω) lying in the interior of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$ can be found such that a trajectory with the initial values $(x_i, y_i, v_r)_0$ and any set of values of (ω) lies in (H) for $t_0 \leq t \leq \tau + h$; for $t = \tau + h$ its u-values satisfy $\sum_{r=1}^{N} u^2_r > d^2/4$, while for $t = \tau - h$ the inequality $\sum_{r=1}^{N} u^2_r < d^2/4$ is satisfied.

Hence to each set of values of (ω) corresponds a value $\bar{\tau}$, and $\bar{\tau} > \tau - h$ since $\sum_{r=1}^{N} u^2_r < d^2/4$, $t_0 \leq t \leq \tau - h$. Also, $\bar{\tau} < \tau + h$ since for $t = \tau + h$, $\sum_{r=1}^{N} u^2_r > d^2/4$. The set of values (u) corresponding to a set belonging to (ω) is formed for $t = \bar{\tau}$. Since h is arbitrarily small, it follows that by choosing the neighborhood (ω) sufficiently small the u-coördinates of a trajectory with initial values $(x_i, y_i, v_r)_0$, (u) in (ω) can be made to differ as little as may be desired from their values on the trajectory with initial values $(x_i, y_i, v_r)_0$, $(u)_0$ at the same time; and the u-coördinates of this trajectory in the interval $\tau - h \leq t \leq \tau + h$ differ as little as may be desired from their values for $t = \tau$. Hence a neighborhood of the set $(u)_0$ can be found such that the set (u') which corresponds to a point $(u')_0$ of this neighborhood and the set (u'') which corresponds to a point $(u'')_0$ of the same neighborhood will satisfy the condition $| u''_r - u'_r | < \epsilon$ $(r = 1, \cdots, N)$, where ϵ has been chosen arbitrarily small. The continuity of this correspondence for points $(u)_0$ in the interior of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$ follows; the proof for points on the boundary can be given in the same manner.

---

* The following argument is based on the continuity of the solutions of a differential system with respect to the initial values of the variables; see, for example, Goursat, *Cours d'Analyse*, tome III, Chap. 23, (1915).

Consequently to each point in the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$ corresponds a point $U_1, U_2, \cdots, U_N$ on the boundary, while points on the boundary correspond to themselves, and the coördinates U_r are continuous functions of the variables $u_1, u_2, \cdots, u_N$. But such a correspondence is impossible, as a consequence of a theorem of Bohl.[*]. Hence the theorem.

In the same manner can be proved the theorem: To every set of values (x_i, y_i, u_r) of the region

$$\sum_{i=1}^{M} (x^2_i + y^2_i) \leq 2d^2/9, \quad \sum_{r=1}^{N} u^2_r \leq d^2/4$$

corresponds a set (v) of the region $\sum_{r=1}^{N} v^2_r \leq d^2/4$ such that the trajectory with this set of initial values at $t = t_0$ lies in the region (A, B) for $t < t_0$.

5. On the manifold of the trajectories whose existence has been established.

Suppose a neighborhood (H') determined such that (H') has the properties of both (D') and (H); such a neighborhood can be determined, with $l_1 = l_2 = \cdots = l_M = 1$, and $m_1 = m_2 = \cdots = m_N = m < 1$. Then suppose two trajectories to lie within (H') for $t \geq t_0$; then if in the terminology of § 1, at $t' \geq t_0$.

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) + m^2 \sum_{r=1}^{N} v'^2_r - \sum_{r=1}^{N} u'^2_r < 0;$$

---

[*] *L. c.*, p. 198. On the boundary the functions U_r cannot vanish simultaneously.

from the theorem of § 1 it would follow that $\sum_{r=1}^{N} u'^2_r$ increases indefinitely with t. Consequently for $t \geqq t_0$,

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) + m^2 \sum_{r=1}^{N} v'^2_r \geqq \sum_{r=1}^{N} u'^2_r.$$

Hence the theorem: If two trajectories lie in (H') for $t \geqq t_0$, and if for some $t' \geqq t_0$, the coördinates (x_i, y_i, v_r) coincide, the two trajectories coincide.

If the u-coördinates corresponding to a given set of values of x_i, y_i, v_r considered as functions of these coördinates, are denoted by $[u_1]$, $[u_2]$, $\cdots$, $[u_N]$, the inequality above proves the continuity of these functions. In the same manner can be obtained the theorem: If two trajectories lie in (H') for $t < t_0$ and if for some $t'' \leqq t_0$ the x_i, y_i, u_r coördinates coincide, the two trajectories coincide.

The v-coördinates of a trajectory which lies within (H') for $t < t_0$ will be denoted by $[v_1]$, $[v_2]$, $\cdots$, $[v_N]$; the continuity of these functions follows as before.

6. Trajectories which lie in (H') for all values of t.

Suppose fixed a set of values (x_i, y_i) satisfying (a), (§ 3), and construct the functions

$$u_r - [u_r], \quad v_r - [v_r] \quad r = 1, \cdots, N.$$

These are continuous functions of the variables u_r, v_r in the region

$$(B) \qquad \sum_{r=1}^{N} u^2_r \leqq d^2/4, \quad \sum_{r=1}^{N} v_r{}^2 \leqq d^2/4.$$

and consequently from a theorem of Bohl,* there exists a point on the boundary of (B) for which, unless these functions vanish simultaneously,

$$u_r - [u_r] = -\nu u_r, \quad v_r - [v_r] = -\nu v_r, \quad \nu > 0, \quad r = 1, \cdots, N,$$

consequently

$$\sum_{r=1}^{N} u^2_r = d^2/4 \text{ or } \sum_{r=1}^{N} v^2_r = d^2/4.$$

From

$$(1+\nu) u_r = [u_r], \quad (1+\nu) v_r = [v_r] \quad r = 1, \cdots, N$$

---

* *L. c.*, pp. 199, 200. The theorem of Bohl is proved for a slightly different domain, but the proof is sufficiently general to cover the case considered here, as is immediately evident; instead of his region (G) we have (B), and a corresponding interpretation of his functions $\phi_i (i = 1, \cdots, n)$.

it follows that

$$(1+\nu)^2\sum_{r=1}^{N} u^2_r = \sum_{r=1}^{N} [u_r]^2, \quad (1+\nu)^2\sum_{r=1}^{N} v^2_r = \sum_{r=1}^{N} [v_r]^2$$

and therefore

$$(1+\nu)^2\, d^2/4 = \sum_{r=1}^{N} [u_r]^2 \quad \text{or} \quad (1+\nu)^2\, d^2/4 = \sum_{r=1}^{N} [v_r]^2$$

Since $\sum_{r=1}^{N} [u_r]^2 \leq d^2/4$, $\sum_{r=1}^{N} [v_r]^2 \leq d^2/4$, we have arrived at a contradiction unless the existence is admitted of a set of values for which

$$u_r = [u_r], \quad v_r = [v_r], \quad r = 1, \cdots, N.$$

Consequently we have the theorem: To every set of values of the region

$$\sum_{r=1}^{M} (x^2_i + y^2_i) \leq 2d^2/9$$

corresponds a set of the region

$$\sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4$$

such that the trajectory with these initial values (x_i, y_i), (u_r, v_r) at $t = t_0$ remains in the region (A, B) for all values of the time.

Consider two trajectories which remain in (A, B) for all values of t. From § 3 we obtain the inequalities

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) + m^2 \sum_{r=1}^{N} v'^2_r \geq \sum_{r=1}^{N} u'^2_r$$

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) + m^2 \sum_{r=1}^{N} u'^2_r \geq \sum_{r=1}^{N} v'^2_r \qquad m < 1,$$

Therefore

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) \geq \frac{1-m^2}{2} \sum_{r=1}^{N} (u'^2_r + v'^2_r).$$

Hence if for any value of t, the (x_i, y_i) coördinates coincide, the two trajectories coincide.

Expansion Problems in Connection with the Hypergeometric Differential Equation.*

By Bernhard Paul Reinsch.

I. Introduction.

1. *Statement of the Problem.* The object of this paper is the development of an expansion theory for the generalized hypergeometric functions of Goursat † and Pochhammer,‡ and for the ordinary hypergeometric functions § of Gauss in particular, analogous to the one worked out by Neumann and Gegenbauer for Bessel functions.‖

We first treat the expansion of $f(x)$, an arbitrary analytic function of the complex variable x, in series of ordinary hypergeometric functions,

$$\bar{F}_{k+m}(x) = x^{k+m} F(a + k + m, b, 1 + k + m; x), \quad (m = 0, 1, 2, \cdots),$$

which are particular solutions of

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0,$$

where $\alpha = a$, $\beta = b - k - m$, and $\gamma = 1 - k - m$. The quantities a, b, and k are to be any fixed numbers, real or complex, subject to the sole restriction that k shall not be a negative integer.

This expansion is at once developed in a generalized form, analogous to Gegenbauer's generalization of Neumann's expansion in terms of Bessel functions, since this includes as a special case (i. e. $k = 0$) the analogue of Neumann's expansion. Some important properties of $\bar{F}_{k+m}(x)$ and of $G_{m,k}(t)$, the polynomials associated with $\bar{F}_{k+m}(x)$, are then discussed.

This expansion theory is then extended to the generalized hypergeometric functions in Section III, and to the expansion of functions of several variables in Section IV.

---

* Presented to the American Mathematical Society, April 19, 1924.

† *Ann. de l'École Normale*, Ser. 2, Vol. 12 (1883), p. 261 and p. 395.

‡ *Mathematische Annalen*, Vol. 38 (1891), p. 586.

§ Darboux, in "Sur l'approximation des fonctions de très-grands nombres, etc.", *Journal de Mathématiques*, Ser. 3, Vol. 4 (1878), p. 377, treated a special case. He developed the expansions of analytic functions in terms of Jacobi polynomials, $F(-n, \alpha + n, \gamma; x)$, by the method of Neumann.

‖ See Watson, *Theory of Bessel Functions* (London, 1922), Chapters 9 and 16, where an excellent summary of these results is given. Here will also be found additional references.

II. THE EXPANSION OF AN ARBITRARY ANALYTIC FUNCTION IN TERMS OF SOLUTIONS OF THE ORDINARY HYPERGEOMETRIC DIFFERENTIAL EQUATION OF GAUSS.

2. *Expansion of x^{k+m} in a Series of $\bar{F}_{k+m}(x)$.* To determine the coefficients c_m in the expansion

$$x^{k+m} = \sum_{m=0}^{\infty} c_m \bar{F}_{k+m}(x),$$

we insert in the right member the expansion of $\bar{F}_{k+m}(x)$,

$$\bar{F}_{k+m}(x) = x^{k+m} + \frac{(a+k+m)b}{1.(1+k+m)}\, x^{k+m+1}$$

$$+ \frac{(a+k+m)(a+k+m+1)b(b+1)}{1.2(1+k+m)(2+k+m)}\, x^{k+m+2} + \cdots, \quad (m = 0, 1, 2, \cdots),$$

and then equate coefficients of like powers of x. The resulting expansion,

$$(1) \qquad x^{k+m} = \sum_{s=0}^{\infty} \frac{(a+k+m)_s(-b)_s}{s!\,(1+k+m)_s}\, \bar{F}_{k+m+s}(x),$$

where $(a)_s = a(a+1)(a+2)\cdots(a+s-1)$ and $(a)_0 = 1$, we shall show to be absolutely and uniformly convergent and to represent the function x^{k+m}.

3. *An Expression Dominating* $\bar{F}_{k+m}(x) = x^{k+m}F(a+k+m, b, 1+k+m; x)$. Always assuming k not to be a negative integer, we have

$$| \bar{F}_{k+m}(x) | = | x |^{k+m} | F_{k+m}(x) |$$

$$\leqq | x |^{k+m} \Big[1 + \frac{| a+k+m | . | b |}{| 1+k+m | \ . \ 1} | x |$$

$$+ \frac{| a+k+m | . | a+k+m+1 | . | b | . | b+1 |}{| 1+k+m | . | 1+k+m+1 | . \ 2!} | x |^2 + \cdots \Big],$$

where the series in the brackets converges absolutely and uniformly for all finite values of m and for all values of x such that $| x | \leqq R' < 1$. We shall show that $| F_{k+m}(x) |$ possesses an upper bound M for *all* (positive integral) values of m and for all values of x such that $| x | \leqq R' < 1$. It is possible to choose m large enough $(m = m_2 \geqq m_1)$ so that $R(a+k+m_2) > 0$, $R(1+k+m_2) > 0$, and so that for *every* m_2 equal to or greater than m_1, only one of the following two cases exists. *Either*

$$\frac{| a+k+m_1+\nu |}{| 1+k+m_1+\nu |} \leqq \frac{| a+k+m_2+\nu |}{| 1+k+m_2+\nu |} \leqq 1, \quad (\nu = 0, 1, 2, \cdots),$$

or

$$1 \leqq \frac{|a+k+m_2+\nu|}{|1+k+m_2+\nu|} \leqq \frac{|a+k+m_1+\nu|}{|1+k+m_1+\nu|}, \qquad (\nu = 0, 1, 2, \cdots).$$

In the *first case*, we have

$$|F_{k+m_2}(x)| \leqq [1 + \frac{|b|}{1}|x| + \frac{|b|.(|b|+1)}{2!}|x|^2 + \cdots]$$
$$= [1 - |x|]^{-|b|}, \quad (m_2 \geqq m_1),$$

which certainly has an upper bound M_1, for all values of m_2 and for all values of x such that $|x| \leqq R' < 1$.

In considering the *second case*, we note that the expansion of $F_{k+m_1}(x)$ converges absolutely and uniformly, and hence $|F_{k+m_1}(x)|$ has an upper bound M_2 when $|x| \leqq R' < 1$, where m_1 is a value of m such that $|a+k+m+\nu| \geqq |1+k+m+\nu|$, $(\nu = 0, 1, 2, \cdots)$. But since

$$(2) \quad |F_{k+m_1}(x)| \leqq [1 + \frac{|a+k+m_1|.|b|}{|1+k+m_1|.1}|x|$$
$$+ \frac{|a+k+m_1|.|a+k+m_1+1|.|b|.|b+1|}{|1+k+m_1|.|1+k+m_1+1|.2!}|x|^2 + \cdots] \leqq M_2,$$

and since the replacing of m_1 by m_2, where $m_2 > m_1$, in the series (2) gives a series whose terms are less than or equal to the corresponding terms in (2), it follows that for all values of m greater than m_1, we have

$$|F_{k+m}(x)| \leqq M_2, \qquad (m > m_1).$$

It is evident that for each value of m less than m_1, $|F_{k+m}(x)|$ has an upper bound in the region $|x| \leqq R' < 1$. In this group of upper bounds let M_3 be the largest. Then, letting M be the greatest of the quantities M_1, M_2, and M_3, we have

$$(3) \qquad |F_{k+m}(x)| \leqq |x|^{k+m}M$$

for *all* values of m and for all values of x such that $|x| \leqq R' < 1$.

4. *Convergence of the Expansion for* x^{k+m}. By means of the last inequality, it may be seen that the series in the second member of (1) is dominated by the series

$$(4) \qquad \sum_{s=0}^{\infty}\left|\frac{(a+k+m)_s(-b)_s}{s!\,(1+k+m)_s}\right|.|x|^{k+m+s}M.$$

Since the ratio of the $(s+1)$th term to the sth term approaches $|x|$ as s approaches infinity, we see that (4), and hence (1), is absolutely convergent when $|x| \leqq R' < 1$. After dividing out the factor x^k, we can prove the uniform convergence of the series in (1) throughout the region $|x| \leqq R' < 1$, for then (4) divided by x^k will be dominated by a convergent series of positive constants independent of x. Hence, the uniform convergence follows from the test of Weierstrass.

Then, since the power series expansions of $\bar{F}_{k+m}(x)$ all converge absolutely in the circle $|x| \leqq R' < 1$, and (1) divided by x^k converges uniformly in the same circle, we know from the Weierstrass theorem on double series that the rearrangement in deriving (1) was valid and that x^m, and consequently x^{k+m}, is represented by its expansion in the circle $|x| \leqq R'$.

5. *Expansion of $x^k/(t-x)$ in Terms of $\bar{F}_{k+m}(x)$.* In the expansion

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{x^{k+m}}{t^{m+1}},$$

which is valid for $|x| < |t|$, substitute the appropriate expansion (1) for each power of x:

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{1}{t^{m+1}} \sum_{s=0}^{\infty} \frac{(a+k+m)_s(-b)_s}{s!\,(1+k+m)_s} \bar{F}_{k+m+s}(x). \tag{5}$$

Rearrangement of this repeated series, assuming temporarily that this rearrangement is valid, gives

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \left\{ \sum_{s=0}^{m} \frac{[a+k+m-1]_s(-b)_s}{s!\ [k+m]_s} \frac{1}{t^{m-s+1}} \right\} \bar{F}_{k+m}(x),$$

where $[a]_s = a(a-1)(a-2)\cdots(a-s+1)$ and $[a]_0 = 1$. We now define the polynomial $G_{m,k}(t)$ by the equation

$$G_{m,k}(t) = \sum_{s=0}^{m} \frac{[a+k+m-1]_s(-b)_s}{s!\ [k+m]_s} \cdot \frac{1}{t^{m-s+1}}, \quad (m \geqq 0);$$

$G_{m,k}(t)$ is a polynomial in $1/t$ of degree $m+1$. We then have the expansion

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} G_{m,k}(t) \,.\, \bar{F}_{k+m}(x). \tag{6}$$

To justify the above rearrangement it is sufficient to prove the absolute convergence of the repeated series in (5). But for m large enough $(m > m_1)$ so that $m+1-|k| > 0$, (5) is dominated by

$$\sum_{m=0}^{\infty} \frac{M}{|t|^{m+1}} \sum_{s=0}^{\infty} \left| \frac{(a+k+m)_s \,.\, (-b)_s}{s!\ (1+k+m)_s} \right| \,.\, |x|^{k+m+s}$$

$$\leqq M \sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}} \sum_{s=0}^{\infty} \frac{(|a+k|+m)_s . (|b|)_s}{s!\ (m+1-|k|)_s} . |x|^s$$

$$= M \sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}} F(|a+k|+m, |b|, m+1-|k|; |x|).$$

But by the arguments of Art. 3, $F(|a+k|+m, |b|, m+1-|k|; |x|)$ has an upper bound for all values of m greater than m_1 and for all values of x such that $|x| \leqq R' < 1$. We also surely have an upper bound when $m \leqq m_1$. Hence, letting M' be the greater of these upper bounds, (5) is dominated by

$$M.M'. \sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}},$$

which converges absolutely when $|x| < |t|$, where $|x| \leqq R' < 1$. Consequently (5) is absolutely convergent and the expansion (6) is valid under these conditions.

We can readily establish the uniformity of the convergence of the expansion (6) divided by x^k throughout the regions

$$|t| \geqq R, \qquad |x| \leqq R' \qquad \text{where } R' < R,\ R' < 1.$$

When these inequalities are satisfied, (5), and consequently also (6), after dividing each by x^k, will be dominated by a convergent series of positive constants independent of x and t. Hence, the uniformity of the convergence of (6) divided by x^k follows from the test of Weierstrass.

Writing $\tilde{F}_{k+m}(x) = x^{-k}\bar{F}_{k+m}(x) = x^m F_{k+m}(x)$, this result can be stated as

THEOREM I. *Let x and t be two complex variables such that $|x| \leqq R' < 1$ and $|t| \geqq R$, where $R' < R$; then the expansion*

$$\frac{1}{t-x} = \sum_{m=0}^{\infty} G_{m,k}(t)\tilde{F}_{k+m}(x) \tag{7}$$

is valid, provided k is not a negative integer. If t be a variable point on the contour C formed by the circle $|t| = R$, and x any fixed point in the circle $|x| = R'$, then the expansion converges uniformly with regard to t.

6. *Expansion of $f(x)$ in Terms of $\tilde{F}_{k+m}(x)$.* By means of (7) and Cauchy's integral formula, it is possible to get an expansion of $f(x)$ in a series of $\tilde{F}_{k+m}(x)$, $(m = 0, 1, 2, \cdots)$, $f(x)$ being an arbitrary function of x analytic and one-valued inside and upon the contour C, formed by the circle $|x| = R \leqq 1$. We proceed as follows:

$$(8) \qquad f(x) = 1/2\pi i \int_C \frac{f(t)\,dt}{t-x}$$

$$= 1/2\pi i \int_C \left\{ \sum_{m=0}^{\infty} G_{m,k}(t)\,.\,\tilde{F}_{k+m}(x) \right\} f(t)\,dt,$$

that is,

$$(9) \qquad f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x), \quad \text{where } a_m = 1/2\pi i \int_C G_{m,k}(t) f(t)\,dt.$$

This expansion converges uniformly with regard to x when x is in the circle $|x| = R' < 1$. That is, the expansion

$$x^k f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x)$$

is valid when $|x| \leqq R' < 1$. This gives

THEOREM II. *Let $f(x)$ be a function of x which is analytic and one-valued inside and upon the contour C formed by the circle $|x| = R \leqq 1$; then the expansion*

$$(10) \qquad x^k f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x), \quad \textit{where } a_m = 1/2\pi i \int_C G_{m,k}(t) f(t)\,dt,$$

is valid when x is in the circle $|x| = R'$, where $R' < R$, provided k is not a negative integer; and the expansion divided by the factor x^k converges uniformly with regard to x when $|x| \leqq R'$.

7. *Expansion of $f(x)$ in Terms of $G_{m,k}(x)$.* If x and t be two complex variables such that $|t| \leqq r$ and $1 > |x| \geqq r'$, where $r < r'$, then it follows from arguments analogous to those in Art. 5 that the expansion

$$(11) \qquad \frac{1}{x-t} = \sum_{m=0}^{\infty} \tilde{F}_{k+m}(t)\,.\,G_{m,k}(x)$$

is valid, provided k is not a negative integer; and if t be a variable point on the contour c formed by the circle $|t| = r$, and x any fixed point either on or outside the circle $|x| = r'$ but inside the circle $|x| = 1$, then the expansion (11) converges uniformly with regard to t.

Now if $f(x)$ is a function of x, one-valued and analytic, outside and upon the contour c, it can be shown by arguments analogous to those in Art. 6 to have an expansion in a series of $G_{m,k}(x)$, namely,

$$(12) \qquad f(x) = \sum_{m=0}^{\infty} a'_m\, G_{m,k}(x); \qquad \text{where } a'_m = 1/2\pi i \int_c \tilde{F}_{k+m}(t) f(t)\,dt,$$

which will be valid and converge uniformly with regard to x when x is either on or outside the circle $|x| = r'$ but within the circle $|x| = 1$, where $r < r'$, provided k is not a negative integer.

8. *Relation to Maclaurin's Expansion.* If the Maclaurin expansion of $f(x)$ is

$$f(x) = \sum_{m=0}^{\infty} b_m x^m,$$

while the expansion of Theorem II is

$$x^k f(x) = \sum_{m=0}^{\infty} a_m \bar{F}_{k+m}(x),$$

it is possible to find the relation between the coefficients b_m and a_m. We see that

$$\begin{aligned} a_m &= 1/2\pi i \int_C G_{m,k}(t) f(t)\, dt \\ &= 1/2\pi i \int_C G_{m,k}(t) \left\{ \sum_{n=0}^{\infty} b_n t^n \right\} dt \\ &= 1/2\pi i \sum_{n=0}^{\infty} b_n \int_C t^n \left\{ \sum_{\nu=0}^{m} \frac{[a+k+m-1]_\nu \ (-b)_\nu}{\nu!\ [k+m]_\nu} \cdot \frac{1}{t^{m-\nu+1}} \right\} dt \end{aligned}$$

$$(13) \qquad \begin{cases} a_m = \displaystyle\sum_{n=0}^{m} \frac{[a+k+m-1]_n (-b)_n}{n!\ [k+m]_n} \ b_{m-n}, \quad (m = 0, 1, 2, \cdots), \\ a_0 = b_0. \end{cases}$$

These formulae enable us to construct the expansion in Theorem II of a given function when the Maclaurin expansion of that function is given. It is clear that the expansion of a given function in a series of ordinary hypergeometric functions will have simple coefficients whenever the expressions in (13) take on simple forms.

9. *Analogue of Laurent's Theorem.* Let $f(x)$ be an analytic and one-valued function of x in the ring-shaped region defined by the inequalities

$$0 < r \leqq |x| \leqq R \leqq 1.$$

Let C and c be the contours formed by the circles $|x| = R$, $|x| = r$, and C' and c' the contours formed by the circles $|x| = R'$, $|x| = r'$, where $R > R'$ and $r < r'$.

Then, if x is any point in the ring-shaped region defined by the inequalities

$$r' \leqq |x| \leqq R',$$

we have from Cauchy's theorem that

$$(14)\qquad f(x)=1/2\pi i\int_C \frac{f(t)\,dt}{t-x}+1/2\pi i\int_c \frac{f(t)\,dt}{x-t},$$

where the integrations along the contours C and c are both taken in a counterclockwise direction. For the first integral in (14), we have $|x|\leqq R'$ and $|t|=R$, where $R'<R\leqq 1$; hence, Theorem I applies and gives

$$\frac{1}{t-x}=\sum_{m=0}^{\infty} G_{m,k}(t)\,.\tilde{F}_{k+m}(x).$$

Similarly for the second integral, where $|t|=r$, $|x|\geqq r'$, and $r<r'$, we have from (11)

$$\frac{1}{x-t}=\sum_{m=0}^{\infty} \tilde{F}_{k+m}(t)\,.\,G_{m,k}(x).$$

Substituting these two expressions in (14), we have

$$f(x)=1/2\pi i\int_C\left\{\sum_{m=0}^{\infty} G_{m,k}(t)\,.\tilde{F}_{k+m}(x)\right\}f(t)\,dt$$
$$+1/2\pi i\int_c\left\{\sum_{m=0}^{\infty} \tilde{F}_{k+m}(t)\,.\,G_{m,k}(x)\right\}f(t)\,dt.$$

Consequently, the expansion of $f(x)$ is in the form

$$f(x)=\sum_{m=0}^{\infty} a_m\tilde{F}_{k+m}(x)+\sum_{m=0}^{\infty} a'_m G_{m,k}(x),$$

where $a_m=1/2\pi i\int_C G_{m,k}(t)f(t)\,dt,\quad a'_m=1/2\pi i\int_c \tilde{F}_{k+m}(t)f(t)\,dt,$

and is valid when $r'\leqq|x|\leqq R'$. This result may be stated as

THEOREM III. *If $f(x)$ is an analytic and one-valued function of x in the ring-shaped region defined by the inequalities*

$$0<r\leqq|x|\leqq R\leqq 1,$$

the expansion

$$(15)\qquad f(x)=\sum_{m=0}^{\infty} a_m\tilde{F}_{k+m}(x)+\sum_{m=0}^{\infty} a'_m G_{m,k}(x),$$

where

$$a_m=1/2\pi i\int_C G_{m,k}(t)f(t)\,dt \quad \textit{and}\quad a'_m=1/2\pi i\int_c \tilde{F}_{k+m}(t)f(t)\,dt,$$

is valid and converges uniformly with regard to x when

$$r' \leqq |x| \leqq R',$$

where $r < r'$ *and* $R' < R$, *provided* k *is not equal to a negative integer.*

10. *Relation of the Analogue to the Actual Laurent Expansion.* If the Laurent expansion of $f(x)$ in the region

$$r' \leqq |x| \leqq R'$$

is

$$f(x) = \sum_{m=0}^{\infty} b_m x^m + \sum_{m=1}^{\infty} b'_m x^{-m},$$

we can find the relation between the coefficients b_m, b'_m of this expansion and a_m, a'_m of (15). As before, in Art. 8, we find the value of a_m as given by the formulae (13); and we see that

$$\begin{aligned} a'_m &= 1/2\pi i \int_c \tilde{F}_{k+m}(t) f(t)\, dt \\ &= 1/2\pi i \int_c \tilde{F}_{k+m}(t) \left\{ \sum_{n=1}^{\infty} b'_n/t^n \right\} dt \\ &= 1/2\pi i \sum_{n=1}^{\infty} b'_n \int_c t^{-n} \quad \tilde{F}_{k+m}(t)\, dt \\ &= 1/2\pi i \sum_{n=1}^{\infty} b'_n \int_c t^{-n} \quad \left\{ \sum_{\nu=0}^{\infty} \frac{(a+k+m) \quad (b)_\nu}{\nu!\ (1+k+m)_\nu} t^{m+\nu} \right\} dt. \end{aligned}$$

Hence, we have

$$(16) \qquad \left\{ \begin{aligned} a'_m &= \sum_{n=0}^{\infty} \frac{(a+k+m)_n (b)_n}{n!\ (1+k+m)_n} b'_{m+n+1}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (m = 0, 1, 2, \cdots). \\ a_m &= \sum_{n=0}^{m} \frac{[a+k+m-1]_n (-b)_n}{n!\ [k+m]_n} b_{m-n}. \end{aligned} \right.$$

11. *Recurrence Formulae for* $\bar{F}_{k+n}(x)$. Let the symbols F_0, F_1, and F_2, respectively, denote $F(a, b, c; x)$, $F(a+1, b, c+1; x)$, and $F(a+2, b, c+2; x)$, and the symbol F'_k the derivative of F_k with respect to x. From the four relations given by Gauss,

$$\begin{aligned} &\text{(AA)} \quad cF(a, b+1, c; x) - cF(a+1, b, c; x) \\ &\qquad\qquad -(a-b)xF(a+1, b+1, c+1; x) = 0, \\ &\text{(BB)} \quad c(c+1)F(a, b+1, c+1; x) - c(c+1)F(a, b, c; x) \\ &\qquad\qquad -a(a-b)xF(a+1, b+1, c+2; x) = 0, \\ (17)& \\ &\text{(CC)} \quad cF(a, b+1, c; x) - cF(a, b, c; x) \\ &\qquad\qquad -axF(a+1, b+1, c+1; x) = 0, \end{aligned}$$

$$\text{(DD)}\quad \begin{aligned}&[c-2b+(b-a)x]F(a,b,c;x)\\ &\qquad +b(1-x)F(a,b+1,c;x)-(c-b)F(a,b-1,c;x)=0\end{aligned}$$

one can easily derive the following formulae, from which in turn we can derive recurrence relations for $\bar{F}_{k+n}(x)$.

$$(18)\quad \begin{array}{ll}
\text{(A)} & c(c+1)F_0-(c+1)[c+(a-b+1)x]F_1 \\
 & \qquad\qquad +(a+1)(c-b+1)xF_2=0, \\
\text{(B)} & acF_0-c(1-x)F'_0+a(b-c)F_1=0, \\
\text{(C)} & c(a+1)(c+1)F_0-(c+1)(1-x)[c+(a-b+1)x]F'_1 \\
 & \qquad\qquad +(a+1)(b-c-1)(c-bx)F_2=0, \\
\text{(D)} & cF'_0-abF_1-axF'_1=0, \\
\text{(E)} & c(c+1)(c+1-bx)F'_0 \\
 & \quad -ab(c+1)[c+1+(a-b+1)x]F_1 \\
 & \qquad\qquad -a(a+1)(c-b+1)x^2F'_2=0, \\
\text{(F)} & cF_0-(c-bx)F_1-x(1-x)F'_1=0, \\
\text{(G)} & abcF_0-c(c-bx)F'_0-a(b-c)xF'_1=0, \\
\text{(H)} & c(c+1)(1-x)F'_0-a(c+1)[b+(a-b+1)x]F_1 \\
 & \qquad\qquad +a(a+1)(c-b+1)xF_2=0.
\end{array}$$

To derive (A): replace b by $b-1$ in (AA); a, b, respectively, by $a+1$, $b+1$ in (BB); and eliminate $F(a+1,\ b-1,\ c;\ x)$ between the two equations. The relation (B) is derived from the three relations (CC), (DD), and (CC) with b replaced by $b-1$, by eliminating $F(a,\ b+1,\ c;\ x)$ and $F(a,\ b-1,\ c;\ x)$. To get (C), replace a, c by $a+1$, $c+1$ in (B) and eliminate F_1 between the result and (A). We get (D) from (CC) by replacing a, c respectively, by $a+1$, $c+1$.

To get (E): replace a, c by $a+1$, $c+1$ in (CC); a, b, c by $a+1$, $b+1$, $c+1$ in (A); and eliminate $F(a+2,\ b+1,\ c+2;\ x)$. Now get (F) from (B) and (D) by eliminating F'_0; and (G) from (B) and (D) by eliminating F_1. Finally, eliminate F_0 between (A) and (B) to get (H).

The corresponding recurrence relations for $F_{k+n}=F_{k+n}(x)=F(a+k+n,\ b,\ 1+k+n;\ x)$ are derived from the formulae (18) by replacing a, b, c, respectively, by $a+k+n-1$, b, $k+n$. We shall not take the space to write down these resulting formulae for $F_{k+n}(x)$, but we shall refer to them as (A′), (B′), · · ·, and (H′).

The recurrence formulae for $\bar{F}_{k+n}=\bar{F}_{k+n}(x)=x^{k+n}F(a+k+n,\ b,\ 1+k+n;\ x)$ corresponding to those for F_{k+n}, and derived from them by means of the relations

$$(19)\qquad F_{k+n} = x^{-k-n}\bar{F}_{k+n},\quad F'_{k+n} = x^{-k-n}\bar{F}'_{k+n} - (k+n)x^{-k-n-1}\bar{F}_{k+n},$$

are

$$\begin{aligned}
&(\alpha)\quad (k+n)(k+n+1)x\bar{F}_{k+n-1}\\
&\qquad\qquad -(k+n+1)[k+n+(a-b+k+n)x]\bar{F}_{k+n}\\
&\qquad\qquad\qquad +(a+k+n)(1-b+k+n)\bar{F}_{k+n+1} = 0,\\
&(\beta)\quad (k+n)(k+n-1+ax)\bar{F}_{k+n-1} - (k+n)x(1-x)\bar{F}'_{k+n-1}\\
&\qquad\qquad\qquad +(a+k+n-1)(b-k-n)\bar{F}_{k+n} = 0,\\
&(\gamma)\quad (k+n)(k+n+1)(k+n+ax)\bar{F}_{k+n-1}\\
&\qquad\qquad -(k+n+1)(1-x)[k+n+(a-b+k+n)x]\bar{F}'_{k+n}\\
&\qquad\qquad\qquad -(a+k+n)(1-b+k+n)(k+n-b)\bar{F}_{k+n+1} = 0,\\
&(\delta)\quad (k+n)x\bar{F}'_{k+n-1} + a(k+n-b)\bar{F}_{k+n}\\
&\qquad\qquad\qquad -(k+n-1+ax)\bar{F}'_{k+n} = 0,
\end{aligned}$$

(20)

$$\begin{aligned}
&(\epsilon)\quad (k+n)\bar{F}_{k+n-1} + (b-k-n)\bar{F}_{k+n} - (1-x)\bar{F}'_{k+n} = 0,\\
&(\zeta)\quad a(k+n)\bar{F}_{k+n-1} + (k+n)x\bar{F}'_{k+n-1} - (a+k+n-1)\bar{F}'_{k+n} = 0,\\
&(\eta)\quad (k+n)(k+n+1)x^2\bar{F}'_{k+n-1}\\
&\qquad\qquad + a(k+n+1)[k+n-1+(a-b+k+n)x]\bar{F}_{k+n}\\
&\qquad\qquad\qquad -(a+k+n)(k+n-1+ax)\bar{F}'_{k+n+1} = 0,\\
&(\theta)\quad (k+n)(k+n+1)x^2(1-x)\bar{F}'_{k+n-1}\\
&\qquad -(k+n+1)\{(k+n)(k+n-1) + a(k+n-1+b)x\\
&\qquad\qquad\qquad + a(a-b+k+n)x^2\}\bar{F}_{k+n}\\
&\qquad\qquad +(a+k+n)(1-b+k+n)(k+n-1+ax)\bar{F}_{k+n+1} = 0.
\end{aligned}$$

Relations (α), (β), and (ϵ) follow directly from (A′), (B′), and (F′). To get (γ), transform (C′) by means of (19) and eliminate $\bar{F}_{k+n}$ between this result and (α). To get (δ), transform (D′) by means of (19) and eliminate $\bar{F}_{k+n-1}$ between this and (β). To get (ζ), eliminate $\bar{F}_{k+n}$ between (δ) and (ϵ). To get (η), replace n by $n+1$ in (ζ) and eliminate $\bar{F}'_{k+n}$ between this and (δ). Finally, (θ) is derived from (α) and (β) by eliminating $\bar{F}_{k+n-1}$.

12. *Function Dominating* $G_{m,k}(t)$. It is possible to construct a simple dominating function for the polynomial $G_{m,k}(t)$ which was defined as the coëfficient of $\bar{F}_{k+m}(x)$ in the expansion (6). Assuming throughout that k is not a negative integer, we have

$$|G_{m,k}(t)| \leqq \frac{1}{|t|^{m+1}} \sum_{s=0}^{m} \left|\frac{[a+k+m-1]_s}{[k+m]_s}\right| \frac{(|b|)_s}{s!} |t|^s.$$

By means of the asymptotic formula for the gamma function, it can be shown that for m large enough $(m \geqq m_1)$ the coefficients

$$\left| \frac{(a+k+m-1)(a+k+m-2)\cdots(a+k+m-s)}{(k+m)(k+m-1)\cdots(k+m-s+1)} \right|, \qquad (s=1, 2, \cdots, m),$$

remain less than $M_1 m^{|a-1|}$ for all values of $m > m_1$, and for all values of a, where M_1 is some positive constant. Then

$$\begin{aligned} |\, G_{m,k}(t) \,| &< M_1 \frac{m^{|a-1|}}{|\, t \,|^{m+1}} \sum_{s=0}^{m} \frac{(|\, b \,|)_s}{s\,!} \,|\, t \,|^s \\ &< M_1 \frac{m^{|a-1|}}{|\, t \,|^{m+1}} \sum_{s=0}^{\infty} \frac{(|\, b \,|)_s}{s\,!} \,|\, t \,|^s = M_1 \frac{m^{|a-1|}}{|\, t \,|^{m+1}} [1 - |\, t \,|]^{-|b|}. \end{aligned}$$

But when $|\, t \,| \leqq R'' < 1$, the quantity $[1 - |\, t \,|]^{-|b|}$ has an upper bound M_2 for all values of m; and surely for all values of m less than m_1, we have

$$|\, G_{m,k}(t) \,| < \frac{M_3}{|\, t \,|^{m+1}}, \text{ where } M_3 \text{ is some positive constant.}$$

Let M_4 be some positive constant such that $M_3 > M_4 m^{|a-1|}$, where $m < m_1$. Let M' be the greater of the quantities $(M_1 . M_2)$ and M_4; then

$$(21) \qquad |\, G_{m,k}(t) \,| < M' \frac{m^{|a-1|}}{|\, t \,|^{m+1}}$$

for all values of m, and for all values of t such that $|\, t \,| \leqq R'' < 1$. This expression and the corresponding one for $|\, \bar{F}_{k+m}(x) \,|$, of Art. 3, enable us to dominate the expansion (6) by the series

$$\sum_{m=0}^{\infty} \frac{|\, x \,|^{k+m}}{|\, t \,|^{m+1}} m^{|a-1|} . M . M'.$$

By the ratio test, this series, and consequently also (6), is seen to be absolutely convergent when $|\, x \,| < |\, t \,| \leqq R'' < 1$. Furthermore, the uniform convergence of (6) divided by the factor x^k can now be established throughout the regions

$$|\, x \,| \leqq R', \quad |\, t \,| \geqq R, \quad \text{where } R' < R \leqq R'' < 1, \quad |\, t \,| \leqq R'' < 1;$$

for then (6) is dominated by

$$\sum_{m=0}^{\infty} \frac{(R')^m}{R^{m+1}} m^{|a-1|} M . M',$$

which is an absolutely convergent series of positive constants. Hence, the uniform convergence of (6) divided by x^k follows from the test of Weierstrass, provided that k is not equal to a negative integer.

13. *Recurrence Formulae for the Polynomials* $G_{n,k}(t)$. We shall now derive a few recurrence relations satisfied by $G_{n,k} = G_{n,k}(t)$, the symbol $G'_{n,k}$ denoting the derivative of $G_{n,k}$ with respect to t:

$$
\begin{aligned}
&(1-a-k-n)(k-b+n)G_{n-1,k} \\
&\qquad +(k+n)[n+1+(a+k-2)t]G_{n,k} \\
&\qquad\qquad +(k+n)t(1-t)G'_{n,k} = h_n, \\
&(a+k+n-1)(k-b+n)tG'_{n-1,k} \\
&\qquad -(k+n)[(n+1)k+(b-n)(a+k-2)t]G_{n,k} \\
&\qquad\qquad -(k+n)t[k+(n-b)t]G'_{n,k} = (n-b)h_n, \\
&[k+(n-b)t]G_{n-1,k} - t(1-t)G'_{n-1,k} - (k+n)tG_{n,k} \\
&\qquad\qquad = -\frac{h_n}{a+k+n-1}, \\
&(a+k+n-1)(k-b+n)G_{n-1,k} \\
&\qquad -(k+n)[k+n+1+(a+k-b+n-1)t]G_{n,k} \\
&\qquad\qquad +(k+n)(k+n+1)tG_{n+1,k} = -\frac{b+1}{n+1}h_n,
\end{aligned}
\tag{22}
$$

$$
\text{where } h_n = \frac{\Gamma(a+k+n)\Gamma(-b+n)\Gamma(1+k)}{\Gamma(a+k-1)\Gamma(-b)\Gamma(k+n)n!},
$$

these formulae being valid for $n \geqq 1$. It may be well to note that for $n = 1$, these relations may be put into different forms because of the existence of two-term relations connecting $G_{0,k}$, $G'_{0,k}$, $G_{1,k}$, and $G'_{1,k}$.

To derive the first formula in (22), we note that $G_{n,k}(t)$ is equal to the first $n+1$ terms in the expansion of $t^{-n-1}F(1-a-k-n, -b, -k-n; t)$ in powers of t. Replacing, respectively, $a+k+n-1$, b, and $k+n$ in (18-B′) by $1-a-k-n$, $-b$, and $-k-n$, we have

$$
\begin{aligned}
&(a+k+n-1)(k+n)F(1-a-k-n, -b, -k-n; t) \\
&\quad +(k+n)(1-t)F'(1-a-k-n, -b, -k-n; t) \\
&\quad +(1-a-k-n)(k-b+n) \\
&\qquad\qquad \times F(2-a-k-n, -b, 1-k-n; t) = 0.
\end{aligned}
$$

If in the first member of this equation we replace $F(1-a-k-n, -b, -k-n; t)$, $F'(1-a-k-n, -b, -k-n; t)$, and $F(2-a-k-n, -b, 1-k-n; t)$, respectively, by $t^{n+1}G_{n,k}(t)$, $[t^{n+1}G'_{n,k}(t) + (n+1)t^n \cdot G_{n,k}(t)]$, and $t^nG_{n-1,k}(t)$, we shall have the first recurrence relation in (22) with the second member, h_n, still undetermined. Its value, however, is readily found by substituting the polynomials $G_{n-1,k}$, $G_{n,k}$, and $G'_{n,k}$ into the given relation.

To derive the second formula in (22), we get in similar fashion a recurrence relation, except for undetermined second member, connecting $G_{n-1,k}$, $G'_{n-1,k}$, $G_{n,k}$, and $G'_{n,k}$, from (18-D′) and eliminate $G_{n-1,k}$ between this and the first formula in (22). The second member is then determined in the same fashion as above.

Now the third result in (22) readily follows from the first two by elimination of $G'_{n,k}$, and the last one is derived by replacing n by $n+1$ in the third and then eliminating $G'_{n,k}$ between this result and the first relation.

14. *The Differential Equation Satisfied by* $G_{n,k}(t)$. By differentiating the first recurrence relation in (22) and eliminating $G'_{n-1,k}$ between this result and the second relation in (22), we get the differential equation satisfied by $G_{n,k}(t)$:

$$(23)\quad t(1-t)y'' + [2-k+n+(a+b+k-n-4)t]y' \\ -\left[\frac{(n+1)k}{t} + (a+k-2)(b-1-n)\right] y = \frac{n-b}{k+n}\cdot\frac{h_n}{t},$$

where $$h_n = \frac{\Gamma(a+k+n)\Gamma(-b+n)\Gamma(1+k)}{\Gamma(a+k-1)\Gamma(-b)\Gamma(k+n)\, n!}.$$

15. *Orthogonality Properties.* We shall now obtain the integral formulae

$$(24)\quad \begin{aligned} &\int_L \tilde{F}_{k+m}(x)\,.\,\tilde{F}_{k+n}(x)\,dx = 0, && (m = n \text{ and } m \neq n),\\ &\int_L G_{m,k}(x)\,.\,G_{n,k}(x)\,dx = 0, && (m = n \text{ and } m \neq n),\\ &\int_L \tilde{F}_{k+n}(x)\,.\,G_{n,k}(x)\,dx = 2\pi i\kappa, && (L \text{ enclosing origin}),\\ &\qquad\qquad = 0, && (L \text{ not enclosing origin}),\\ &\int_L \tilde{F}_{k+m}(x)\,.\,G_{n,k}(x)\,dx = 0, && (m \neq n),\end{aligned}$$

where the integration is in a positive direction along L, any closed contour whatever, not passing through the origin and lying within the circle $|x| = 1$. Since $\tilde{F}_{k+m}(x)$ is single-valued and uniformly continuous in the region enclosed by L, the first formula follows immediately from Cauchy's theorem. Since the residue of $G_{m,k}(x)\,.\,G_{n,k}(x)$ at $x = 0$ is zero, the second follows readily. We also have

$$\int_L \tilde{F}_{k+n}(x)\,.\,G_{n,k}(x)\,dx = \int_L dx/x + \sum_{\nu=0}^{\infty}\int_L a_\nu x^\nu dx,$$

which is equal to $2\pi i\kappa$ or zero according as L does or does not enclose the origin, where κ is the excess of the number of positive circuits of the contour around the origin over the number of negative circuits.

In proving the fourth formula, we note that it is true when $m > n$ because the expansion of the integrand in x^{-1} then contains no term in x^{-1}. When $m < n$, say $m = n - s$, $(s = 1, 2, 3, \cdots, n)$, the residue of the integrand at $x = 0$ is

$$2\pi i \, \frac{(a+k+n-1)(a+k+n-2)\cdots(a+k+n-s)}{s!\,(k+n)(k+n-1)\cdots(k+n-s+1)}$$
$$\times\{[b+(-b)]\cdot[(b+1)+(-b+1)]\cdots$$
$$\cdot[(b+s-1)+(-b+s-1)]\},$$

which is zero for all values of s because of the zero factor $[b+(-b)]$. Hence, the fourth formula will be true whether or not L encloses the origin.

By means of these integral formulae (24), the expansions of a function $f(x)$ in series of the forms (9), (12), and (15) can be obtained directly, assuming $f(x)$ to be analytic in the circle $|x| = R \leqq 1$. For example, if we assume $f(x)$ to be expanded in a series of the form

$$(9) \qquad f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x),$$

the coefficients a_m are readily determined by means of the third and fourth formulae in (24). For, if we multiply both members of the equation (9) by $G_{m,k}(x)$ and then integrate each term with respect to x along a contour C enclosing the origin and lying within the unit circle, we have:

$$\int_C G_{m,k}(x) f(x)\, dx = a_m \int_C G_{m,k}(x) \,.\, \tilde{F}_{k+m}(x)\, dx,$$

since by the fourth relation in (24) all the remaining integrals in the second member vanish. We, therefore, have the value of any coefficient a_m in (9), namely:

$$a_m = 1/2\pi i \int_C G_{m,k}(t) f(t)\, dt.$$

Similarly, the coefficients in the expansion (15) are directly determined by means of all four formulae in (24).

16. *The Expansion in Special Cases. When* $k = 0$, the above developed expansion theory, Arts. 2 to 15 inclusive, will hold in every respect, expansions now being in terms of

$$\tilde{F}_n(x) = x^n F(a+n,\ b,\ 1+n;\ x).$$

We may note that there will be no restrictions on a or b.

It is interesting to notice that for $k = 0$ the differential equation (23) satisfied by $G_{n,0}(t)$, is a non-homogeneous differential equation having $(n - b)h_n/t(k + n)$ in the second member, which in turn is a hypergeometric differential equation when the second member is zero. This second member is zero either when $a = 1$ or when $b = 0$.

When $k = 0$ and $a = 1$, our expansions are in terms of

$$\bar{F}_n(x) = x^n F(1 + n,\ b,\ 1 + n;\ x) = x^n(1 - x)^{-b}$$

and

$$G_n(t) = t^{-n-1} \sum_{\nu=0}^{n} \binom{b}{\nu} (-t)^{\nu},$$

where $\bar{F}_n(x)$ and $G_n(t)$ are both solutions of hypergeometric differential equations simply related to each other, namely:

$$x(1 - x)y'' + [1 - n - (b - n + 2)x]y' - (b - n)y = 0,$$
$$t(1 - t)y'' + [2 + n - (-b + n + 3)t]y' - (n + 1 - b)y = 0.$$

Finally, when $k = 0$ and $b = 0$, the expansions are in terms of $\bar{F}_n(x) = x^n F(1 + n,\ 0,\ 1 + n;\ x) = x^n$ and $G_n(t) = t^{-n-1}$, again both being solutions of hypergeometric differential equations. We see, then, that the expansion of Theorem II reduces in this special case to the Cauchy-Taylor expansion, namely:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{where } a_n = 1/2\pi i \int_C \frac{f(t)\,dt}{t^{n+1}}\ .$$

Similarly, the expansion of Theorem III reduces to the ordinary Laurent expansion:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a'_n x^{-n-1},$$

where $$a_n = 1/2\pi i \int_C \frac{f(t)\,dt}{t^{n+1}}\ , \quad \text{and } a'_n = 1/2\pi i \int_c t^n f(t)\,dt.$$

There exists no expansion of $\tilde{F}_{k+m}(x)$ in terms of other hypergeometric functions, $\tilde{F}_{k+m+s}(x)$, $(s = 1, 2, 3, \cdots)$; but replacing n by $n + 1$ in the recurrence relation (20-a) and by applying this relation repeatedly, we can get an expansion of $\tilde{F}_{k+m}(x)$ in terms of $x^{-s-1}\tilde{F}_{k+m+2s+1}(x)$, namely:

$$\tilde{F}_{k+m}(x) = \sum_{s=0}^{\infty} \gamma_s [k + m + 2s + 1 + (a - b + k + m + 2s + 1)x] \times \frac{(2)^{2s}}{x^{s+1}} \tilde{F}_{k+m+2s+1}(x),$$

$$\text{where } \gamma_s = (-1)^s \; \frac{\left(\frac{a+k+n-1}{2} + 1\right.}{(k +}$$

By showing that $\underset{s\to\infty}{L} R_s \to 0$, where R_s is the
we can prove this expansion to be valid when |

III. EXPANSIONS IN TERMS OF GENERALIZED

17. *General Remarks.* The entire expansi
now to be extended to the generalized hypergeon
and Pochhammer,† which satisfy a homogeneou
of order $q + 1$, two of whose particular integrals

$$ {}_pF_q(a_1, a_2, \cdots, a_p; \rho_1, \rho_2, \cdots, \rho_q; x)$$

$$= \sum_{s=0}^{\infty} \frac{(a_1)_s \cdot (a_2)_s \cdot}{(\rho_1)_s \cdot (\rho_2)_s \cdot}$$

and

$$x^{1-\rho_1}{}_pF_q(a_1 - \rho_1 + 1, a_2 - \rho_1 + 1, \cdots, a_p -$$

$$2 - \rho_1, \rho_2 - \rho_1 + 1, \rho_3 - \rho_1 -$$

$$(25) \quad = x^{1-\rho_1} \sum_{s=0}^{\infty} \frac{(a_1 - \rho_1 + 1)_s \cdot (a_2 - \rho_1 + 1)}{(2 - \rho_1)_s \cdot (\rho_2 - \rho_1 + 1)_s \cdot (\rho_3 -}$$

$$\times \quad \frac{x^s}{s!}, \quad (p \geqq$$

where $(a)_s = a(a+1)(a+2) \cdots (a+s-$
$(a_p - \rho_1 + 1)_s = 1$ for $p = 0$. We may note th
tial equation depends upon the number q and no
These series all diverge when $p > q + 1$.
separately the two general cases: (1) $p = q + 1$
$p = q + 1$, the differential equation has two fi
and 1, and the series in (25) converges with
However, *when* $p < q + 1$, the differential equati
larity, $x = 0$, and the series in (25) converges f

18. *Expansions in Terms of Generalized*
Valid Within the Unit Circle: Case, $p = q + 1$.
tion of the complex variable x is now to be expa

---

[*] *Loc. cit.*

[†] *Loc. cit.*

$$(26)\qquad \bar{\mathfrak{F}}_{k+m}(x) = x^{k+m}\mathfrak{F}_{k+m}(x)$$
$$= x^{k+m}\sum_{s=0}^{\infty}\frac{(a_1+k+m)_s.(a_2+k+m)_s\cdots(a_{p-1}+k+m)_s.(a_p)_s}{(1+k+m)_s.(\rho_2+k+m)_s\cdots(\rho_{p-1}+k+m)_s\quad s!}$$
$$(m=0,\,1,\,2,\,\cdots),\quad (p \geqq 2,\, \rho_1 = 1),$$

which is the form that the particular integral (25) changes into wh replace q, ρ_1, and a_p, respectively, by $p-1$, $1-k-m$, and a_p-k. Here $a_1, a_2, \cdots, a_p, \rho_2, \rho_3, \cdots, \rho_{p-1}$, and k are any fixed numbers, r complex, except that none of the quantities $1+k$, ρ_2+k, ρ_3+k, $\rho_{p-2}+k$, and $\rho_{p-1}+k$ is permitted to be a negative integer or zero. expansions in terms of $\bar{F}_{k+m}(x) = x^{k+m}{}_2F_1(a+k+m,\, b;\, 1+k+m;$ investigated in Section II, are a special case of this, namely, when $p=2$.

A dominating expression for $\bar{\mathfrak{F}}_{k+m}(x)$ can be developed by means o argument similar to that used in Art. 3 in connection with $\bar{F}_{k+m}(x)$. $\mathfrak{F}_{k+m}(x)$ can be shown to have an upper bound M for all values of m and all values of x such that $|x| \leqq R' < 1$, since the series for $\mathfrak{F}_{k+m}(x)$ verges absolutely and uniformly for any finite value of m and for all va of x such that $|x| \leqq R' < 1$. We can choose m large enough $(m \geqq m_1)$ that any particular one of the $p-1$ sets of ratios

$$\frac{|a_1+k+m+\nu|}{|1+k+m+\nu|},\qquad \frac{|a_i+k+m+\nu|}{|\rho_i+k+m+\nu|},\qquad \begin{matrix}(i=2,\,3,\,\cdots,\,p-\\ (\nu=0,\,1,\,2,\,\cdots),\end{matrix}$$

will *either* be $\leqq 1$ for all values of m greater than or equal to m_1, *or* be $\geqq$ for all $m \geqq m_1$, and moreover that these ratios approach 1 monotonically m increases beyond m_1. Let us suppose that σ of these $p-1$ sets are $\leqq$ while the remaining $p-1-\sigma$ sets are $\geqq 1$ when $m \geqq m_1$. Now if in t expansion of $\mathfrak{F}_{k+m_1}(x)$,

$$|\mathfrak{F}_{k+m_1}(x)| \leqq \sum_{s=0}^{\infty}\frac{|a_1+k+m_1|_s.|a_2+k+m_1|_s\cdots|a_{p-1}+k+m_1|_s.|a_p|_s}{|1+k+m_1|_s.|\rho_2+k+m_1|_s\cdots|\rho_{p-1}+k+m_1|_s.s!}|x|$$

we replace each of the ratios in these σ sets of ratios $\leqq 1$ by 1, the series i the second member will be dominated by a series which is absolutely an uniformly convergent when $|x| \leqq R' < 1$; hence, the dominating series, an also $|\mathfrak{F}_{k+m_1}(x)|$, will have an upper bound M_1 for $|x| \leqq R' < 1$. If in thi dominating series we replace m_1 by any value of m (say m_2) which is greate than m_1, we get a new series which is term by term less than the series in m_1. We readily see, therefore, that $|\mathfrak{F}_{k+m_2}(x)|$, $(m_2 > m_1)$, has an upper bound M_2.

Also $|\mathfrak{F}_{k+m}(x)|$ has an upper bound in the region $|x| \leqq R' < 1$ for each value of m less than m_1. Let M_3 be the largest of these. It is evident ıat these results hold equally well when σ is equal to zero or $p-1$. Theı M is the greatest of the quantities M_1, M_2, and M_3, we have

$$27) \qquad | \bar{\mathfrak{F}}_{k+m}(x) | \leqq | x |^{k+m} M$$

ɪr *all* values of m and for all values of x such that $| x | \leqq R' < 1$.

By the same method used (Art. 2) in finding the expansion of x^{k+m} iɪ ːrms of $\bar{F}_{k+m}(x)$, we show that

$$28) \quad x^{k+m}$$

$$= \sum_{s=0}^{\infty} \frac{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_{p-1}+k+m)_s \cdot s!}{(a_1+k+m)_s \cdot (a_2+k+m)_s \cdots (a_{p-1}+k+m)_s \cdot (-a_p)_s} \bar{\mathfrak{F}}_{k+m+s}(x)$$

y means of the expression in (27) dominating $| \bar{\mathfrak{F}}_{k+m}(x) |$, this expansioɪ readily proved to be absolutely convergent when $| x | \leqq R' < 1$; and afteɪ .viding out the factor x^k, it is seen to be uniformly convergent wheɪ $x | \leqq R' < 1$.

Similarly as in Art. 5, we have, assuming $| x | < | t |$,

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{x^{k+m}}{t^{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{1}{t^{m+1}} \sum_{s=0}^{\infty} \frac{(a_1+k+m)_s (a_2+k+m)_s \cdots (a_{p-1}+k+m)_s (-a_p)_s}{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_{p-1}+k+m)_s \cdot s!}$$

$$29) \quad = \sum_{m=0}^{\infty} \mathfrak{g}_{m,k}(t) \cdot \bar{\mathfrak{F}}_{k+m}(x),$$

ıere $\mathfrak{g}_{m,k}(t)$, the coefficient of $\bar{\mathfrak{F}}_{k+m}(x)$, is a polynomial of degree $m+1$ t^{-1}, defined by the equation

$$0) \quad {}_{m,k}(t)$$

$$= \sum_{s=0}^{m} \frac{[a_1+k+m-1]_s \cdot [a_2+k+m-1]_s \cdots [a_{p-1}+k+m-1]_s \cdot (-a_p)_s}{[k+m]_s \cdot [\rho_2+k+m-1]_s \cdots [\rho_{p-1}+k+m-1]_s \cdot s!} \; \frac{1}{t^{m-s+1}},$$

ıere $[a]_s = a(a-1)(a-2) \cdots (a-s+1)$, $[a]_0 = 1$.
ith arguments similar to those used in Art. 5, we justify the rearrangeɜnt of the above repeated series, prove the expansion (29) to be valid and nverge absolutely when $| x | < | t |$ and $| x | \leqq R' < 1$, and prove that ꞉9) divided by x^k converges uniformly when $| x | \leqq R'$; $| t | \geqq R$, where $< R \leqq 1$.

Writing $\tilde{\mathfrak{F}}_{k+m}(x) = x^m \mathfrak{F}_{k+m}(x)$, we can now state for the generalized pergeometric functions (case $p = q + 1$) theorems corresponding to Theɜms I, II, and III, keeping in each the restriction that none of the quantis $1 + k$, $\rho_2 + k$, $\rho_3 + k$, $\cdots$, $\rho_{p-1} + k$ may be a negative integer or zero:

THEOREM IV. *Let x and t be two complex variables such that*

$$|x| \leqq R' \quad \text{and} \quad |t| \geqq R, \quad \text{where} \quad R' < R \leqq 1,$$

then the expansion

$$\frac{1}{t-x} = \sum_{m=0}^{\infty} \mathfrak{G}_{m,k}(t)\, \tilde{\mathfrak{F}}_{k+m}(x) \tag{31}$$

is valid. If t be a variable point on the contour C formed by the circle $|t| = R$, and x any fixed point in the circle $|x| = R'$, then the expansion converges uniformly with regard to t.

THEOREM V. *Let $f(x)$ be a function of x which is analytic and one-valued inside and upon the contour C formed by the circle $|x| = R \leqq 1$, then the expansion*

$$x^k f(x) = \sum_{m=0}^{\infty} a_m\, \bar{\mathfrak{F}}_{k+m}(x), \quad \text{where} \quad a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt, \tag{32}$$

is valid when $|x| \leqq R' < R \leqq 1$; and the expansion divided by the factor x^k converges uniformly with regard to x when $|x| \leqq R'$.

THEOREM VI. *If $f(x)$ is an analytic and one-valued function of x in the ring-shaped region defined by the inequalities*

$$0 < r \leqq |x| \leqq R \leqq 1,$$

the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m\, \tilde{\mathfrak{F}}_{k+m}(x) + \sum_{m=0}^{\infty} a'_m\, \mathfrak{G}_{m,k}(x), \tag{33}$$

where $a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt$ *and* $a'_m = 1/2\pi i \int_c \tilde{\mathfrak{F}}_{k+m}(t) f(t)\, dt$,

is valid and converges uniformly with regard to x when

$$r' \leqq |x| \leqq R',$$

where $r' > r$ and $R' < R$, and where C and c are the contours formed by the circles $|x| = R'$ and $|x| = r'$, respectively.

The formulae corresponding to (13) of Art. 8, and giving the relation between the coefficients b_m of *Maclaurin's expansion,*

$$f(x) = \sum_{m=0}^{\infty} b_m x^m,$$

and the coefficients a_m of the expansion of Theorem V, are

$$34)\quad\begin{cases} a_m = \sum\limits_{n=0}^{m} \dfrac{[a_1+k+m-1]_n\,.\,[a_2+k+m-1]_n\cdots[a_{p-1}+k+m-1]_n\,.\,(-a_p)_n}{[k+m]_n\,.\,[\rho_2+k+m-1]_n\cdots[\rho_{p-1}+k+m-1]_n\;.\;n!} \\ a_0 = b_0, \qquad\qquad (m=0,1,2,\cdots). \end{cases}$$

Similarly, the formulae corresponding to (16) of Art. 10, giving the elation between the coefficients b_m, b'_m of *Laurent's* actual expansion,

$$f(x) = \sum_{m=0}^{\infty} b_m x^m + \sum_{m=1}^{\infty} b'_m x^{-m},$$

nd the expansion of Theorem VI, will be

$$35)\quad\begin{cases} a'_m = \sum\limits_{n=0}^{\infty} \dfrac{(a_1+k+m)_n\,.\,(a_2+k+m)_n\cdots(a_{p-1}+k+m)_n\,.\,(a_p)_n}{(1+k+m)_n\,.\,(\rho_2+k+m)_n\cdots(\rho_{p-1}+k+m)_n\quad n!}\,b'_{m+n+1}, \\ a_m = \sum\limits_{n=0}^{m} \dfrac{[a_1+k+m-1]_n\,.\,[a_2+k+m-1]_n\cdots[a_{p-1}+k+m-1]_n\,.\,(-a_p}{[k+m]_n\,.\,[\rho_2+k+m-1]_n\cdots[\rho_{p-1}+k+m-1]_n\;.\;n!} \\ \qquad\qquad (m=0,1,2,\cdots). \end{cases}$$

Orthogonality properties corresponding to those of Art. 15 hold:

$$36)\quad\begin{aligned} &\int_L \tilde{\mathfrak{J}}_{k+m}(x)\,.\,\tilde{\mathfrak{J}}_{k+n}(x)\,dx = 0, && (m=n \text{ and } m\neq n),\\ &\int_L \mathfrak{G}_{m,k}(x)\,.\,\mathfrak{G}_{n,k}(x)\,dx = 0, && (m=n \text{ and } m\neq n),\\ &\int_L \tilde{\mathfrak{J}}_{k+n}(x)\,.\,\mathfrak{G}_{n,k}(x)\,dx = 2\pi i\kappa, && (L \text{ enclosing origin}),\\ &\qquad\qquad = 0, && (L \text{ not enclosing origin}),\\ &\int_L \tilde{\mathfrak{J}}_{k+m}(x)\,.\,\mathfrak{G}_{n,k}(x)\,dx = 0, && (m\neq n), \end{aligned}$$

here L and κ have the same meanings as in Art. 15.

19. *Expansions in Terms of Generalized Hypergeometric Functions, alid in Any Finite Region: Case* $p < q+1$. Arbitrary functions of x, nalytic in any finite region, are now to be expanded in terms of

$$37)\quad\begin{aligned} \overline{\mathfrak{F}}_{k+m}(x) &= x^{k+m}\,\mathfrak{F}_{k+m}(x)\\ &= x^{k+m}\sum_{s=0}^{\infty}\frac{(a_1+k+m)_s\,.\,(a_2+k+m)_s\cdots(a_p+k+m)_s}{(1+k+m)_s\,.\,(\rho_2+k+m)_s\cdots(\rho_q+k+m)_s}\,.\,\frac{x^s}{s!},\\ &\qquad (m=0,1,2,\cdots),\quad (p\leqq q,\; q\geqq 1,\;\; \rho_1=1), \end{aligned}$$

here $(a_p+k+m)_s = 1$ for $p=0$, which is the form assumed by the particular integral (25) when we replace ρ_1 by $1-k-m$. Again, $a_1, a_2, \cdots,$

$a_p, \rho_2, \rho_3, \cdots, \rho_q$, and κ are any fixed numbers, real or complex, except that none of the quantities $1 + k, \rho_2 + k, \cdots, p_q + k$ is permitted to be a negative integer or zero.

The following dominating expression,

$$|\, \overline{\mathfrak{F}}_{k+m}(x) \,| \leqq |\, x \,|^{k+m} M,$$

where M is a positive constant, valid for all values of m and for all values of x in any given finite region, is readily obtained by employing the arguments of the previous article when $p = q$, and a fortiori when $p < q$. By means of this dominating expression, the expansion

$$(38) \quad x^{k+m} = \sum_{s=0}^{\infty} (-1)^s \frac{(a_1+k+m)_s \cdot (a_2+k+m)_s \cdots (a_p+k+m)_s}{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_q+k+m)_s \quad s!} \overline{\mathfrak{F}}_{k+m+s}(x)$$

obtained as before, is shown to be absolutely convergent for all values of x in any finite region; and after dividing out the factor x^k, it is shown to be uniformly convergent for all values of x in any given finite region.

We again have

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{x^{k+m}}{t^{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{1}{t^{m+1}} \sum_{s=0}^{\infty} (-1)^s \frac{(a_1+k+m)_s \cdot (a_2+k+m)_s \cdots (a_p+k+m)_s}{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_q+k+m)_s \quad s!} \overline{\mathfrak{F}}_{k+m+s}(x)$$

$$(39) \quad = \sum_{m=0}^{\infty} \mathfrak{G}_{m,k}(t) \cdot \overline{\mathfrak{F}}_{k+m}(x), \qquad (p \leqq q),$$

where we define the polynomial

$$(40) \quad \mathfrak{G}_{m,k}(t)$$

$$= \sum_{s=0}^{m} \frac{(-1)^s}{s!} \cdot \frac{[a_1+k+m-1]_s \cdot [a_2+k+m-1]_s \cdots [a_p+k+m-1]_s}{[k+m]_s \cdot [\rho_2+k+m-1]_s \cdots [\rho_q+k+m-1]_s} \cdot \frac{1}{t^{m-s+1}}.$$

The expansion (39) is valid and converges absolutely for all values of x in any given finite region, assuming $|\, x \,| < |\, t \,|$, and converges uniformly when

$$|\, x \,| \leqq R', \quad |\, t \,| \geqq R, \quad \text{where } R' < R.$$

Writing $\overline{\mathfrak{F}}_{k+m}(x) = x^m\, \mathfrak{F}_{k+m}(x)$, we can state the theorems corresponding to Theorems IV, V, and VI, keeping in each the restriction that none of the quantities $1 + k, \rho_2 + k, \rho_3 + k, \cdots, \rho_q + k$ may be a negative integer or zero:

THEOREM VII. *Let x and t be two complex variables such that*

$$|x| \leqq R' \text{ and } |t| \geqq R,$$

where R is any finite number and $R' < R$; then the expansion

$$(41) \qquad \frac{1}{t-x} = \sum_{m=0}^{\infty} \mathfrak{G}_{m,k}(t)\, \tilde{\mathfrak{F}}_{k+m}(x) \ .$$

is valid. If t be a variable point on the contour C formed by the circle $|t| = R$, and x any fixed point in the circle $|x| = R'$, then the expansion converges uniformly with regard to t.

THEOREM VIII. *Let $f(x)$ be a function of x which is analytic and one-valued inside and upon the contour C formed by the circle $|x| = R$, where R is any finite number, then the expansion*

$$(42) \qquad x^k f(x) = \sum_{m=0}^{\infty} a_m\, \tilde{\mathfrak{F}}_{k+m}(x), \quad \textit{where } a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt,$$

*is valid when $|x| \leqq R' < R$; and the expansion divided by the factor x^k converges uniformly with regard to x when $|x| \leqq R'$.**

THEOREM IX. *If $f(x)$ is an analytic and one-valued function of x in the ring-shaped region defined by the inequalities*

$$0 < r \leqq |x| \leqq R,$$

where R is any finite number, the expansion

$$(43) \qquad f(x) = \sum_{m=0}^{\infty} a_m\, \tilde{\mathfrak{F}}_{k+m}(x) + \sum_{m=0}^{\infty} a'_m\, \mathfrak{G}_{m,k}(x),$$

where $a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt,$ *and* $a'_m = 1/2\pi i \int_c \tilde{\mathfrak{F}}_{k+m}(t) f(t)\, dt,$

is valid and converges uniformly with regard to x when

$$r' \leqq |x| \leqq R',$$

where $r < r'$ and $R' < R$, and where C and c are the contours formed by the circles $|x| = R'$ and $|x| = r'$, respectively.

The sets of formulae corresponding, respectively, to (34) and (35) are

---

* This is a slight extension of a theorem derived in a different way by N. Nielsen, *Théorie des fonctions métasphériques* (Paris, 1911), page 165.

$$(44)\quad \begin{cases} a_m = \sum\limits_{n=0}^{\infty} \dfrac{[a_1+k+m-1]_n \cdot}{[k+m]_n \cdot [\rho_2+} \\ a_0 = b_0 \end{cases}$$

and

$$(45)\quad \begin{cases} a'_m = \sum\limits_{n=0}^{\infty} \dfrac{(a_1+k+m)_n \cdot (}{(1+k+m)_n \cdot (\rho_2} \\ \parallel \qquad \parallel \\ a_m = \sum\limits_{n=0}^{m} \dfrac{[a_1+k+m-1]_n \cdot}{[k+m]_n \cdot [\rho_2+} \end{cases}$$

Let L_1 be any finite closed c
we have the following orthogonali

$$(46)\quad \begin{aligned} &\int_{L_1} \tilde{\mathfrak{F}}_{k+m}(x) \cdot \tilde{\mathfrak{F}}_{k+n}(x)\,dx = \\ &\int_{L_1} \mathfrak{G}_{m,k}(x) \cdot \mathfrak{G}_{n,k}(x)\,dx = \\ &\int_{L_1} \tilde{\mathfrak{F}}_{k+n}(x) \cdot \mathfrak{G}_{n,k}(x)\,dx = \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \\ &\int_{L_1} \tilde{\mathfrak{F}}_{k+m}(x) \cdot \mathfrak{G}_{n,k}(x)\,dx = \end{aligned}$$

It is well to remember that a d
sions is obtained by making k eq
where $p < q + 1$ gives in the spe
$q = 1$, an expansion in terms of
relation

$$\overline{\mathfrak{F}}_{0+m}\left(-\frac{x^2}{4}\right) = \left(-\frac{x^2}{4}\right)^m {}_0F_1$$

where $J_m(x)$ is the Bessel function

$$\overline{\mathfrak{F}}_{0+m}\left(-\frac{z}{4}\right) = \frac{m}{(-}$$

Hence, for an arbitrary function
region inclosing the origin, we ha
tions,

$$f(z) = \sum_{m=0}^{\infty} \beta$$

which was first discussed by Nielse

---

* N. Nielsen, *Handbuch der Theorie*

IV. EXPANSIONS OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES.

20. The above developed expansion theory can easily be extended to th expansion of arbitrary analytic functions of several variables in a multip series of hypergeometric functions. For the sake of simplicity, the case two variables only will be developed, and the result then stated for functio of λ variables.

Assume that k is not equal to a negative integer. Let x_1, t_1 and x_2, be two pairs of complex variables in the x_1 and x_2-planes, respectively, suc that

$$\text{(A)}\qquad \begin{array}{lll} |x_1| \leqq r_1, & |t_1| \geqq R_1, & r_1 < R_1 \leqq 1, \\ |x_2| \leqq r_2, & |t_2| \geqq R_2, & r_2 < R_2 \leqq 1; \end{array}$$

then, by Theorem I, the expansions

$$\text{(B)}\qquad \frac{1}{t_1 - x_1} = \sum_{m_1=0}^{\infty} G_{m_1,k}(t_1) . \tilde{F}_{k+m_1}(x_1),$$

$$\text{(C)}\qquad \frac{1}{t_2 - x_2} = \sum_{m_2=0}^{\infty} G_{m_2,k}(t_2) . \tilde{F}_{k+m_2}(x_2)$$

are valid. If t_1 and t_2 are variable points, respectively, on the contours C and C_2 formed by the circles $|t_1| = R_1$ and $|t_2| = R_2$, and if x_1 and x_2 ar any fixed points in the circles $|x_1| = r_1$ and $|x_2| = r_2$, respectively, then th expansions (B) and (C) converge uniformly with regard to t_1 and t_2 re spectively.

Since both expansions are absolutely and uniformly convergent when th onditions in (A) are fulfilled, then their product,

$$(47)\qquad \frac{1}{t_1 - x_1} \cdot \frac{1}{t_2 - x_2} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} G_{m_1,k}(t_1) . G_{m_2,k}(t_2) . \tilde{F}_{k+m_1}(x_1) . \tilde{F}_{k+m_2}(x_2)$$

s absolutely and uniformly convergent under the same conditions.

Now let $f(x_1, x_2)$ be an arbitrary function of the two independent com lex variables x_1 and x_2, which is analytic and one-valued when x_1 and x_2 are espectively, in the regions bounded by the contours C_1 and C_2; then, b auchy's generalized integral formula, we have

$$f(x_1, x_2) = (1/2\pi i)^2 \int_{C_1} \int_{C_2} \frac{f(t_1, t_2)\, dt_2 dt_1}{(t_1 - x_1)(t_2 - x_2)}$$

$$= (1/2\pi i)^2 \int_{C_1} \int_{C_2} \Big[\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} G_{m_1,k}(t_1) . G_{m_2,k}(t_2) . \tilde{F}_{k+m_1}(x_1) . \tilde{F}_{k+m_2}(x_2) \Big]$$

$$\times \frac{f(t_1, t_2)\, dt_2 dt_1}{(t_1 - x_1)(t_2 - x_2)}$$

$$(48) \qquad = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A_{m_1,m_2} \tilde{F}_{k+m_1}(x_1) \cdot \tilde{F}_{k+m_2}(x_2),$$

where
$$A_{m_1,m_2} = (1/2\pi i)^2 \int_{C_1} \int_{C_2} f(t_1, t_2)\, G_{m_1,k}(t_1) \,.\, G_{m_2,k}(t_2)\, dt_2 dt_1.$$

This expansion is valid and converges uniformly with regard to x_1 and x_2 when x_1 and x_2 are, respectively, in the circles $|x_1| = r_1$ and $|x_2| = r_2$.

Since the extension to any number of variables is evident, we shall state the result for λ variables:

THEOREM X. *Let $f(x_1, x_2, \cdots, x_\lambda)$ be a function of $x_1, x_2, \cdots, x_\lambda$, which is analytic and one-valued when $x_1, x_2, \cdots, x_\lambda$ are, respectively, inside and upon the contours $C_1, C_2, \cdots, C_\lambda$ formed by the circles $|x_1| = R_1 \leqq 1$, $|x_2| = R_2 \leqq 1, \cdots, |x_\lambda| = R_\lambda \leqq 1$, then the expansion*

$$(49) \quad f(x_1, x_2, \cdots, x_\lambda) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_\lambda=0}^{\infty} A_{m_1,m_2,\ldots,m_\lambda} \tilde{F}_{k+m_1}(x_1) \,.\, \tilde{F}_{k+m_2}(x_2) \cdots \tilde{F}_{k+m_\lambda}(x_\lambda$$

where

$$A_{m_1,m_2,\ldots,m_\lambda} = (1/2\pi i)^\lambda \int_{C_1} \int_{C_2} \cdots$$
$$\cdot \int_{C_\lambda} f(t_1, t_2, \cdots, t_\lambda) \,.\, G_{m_1,k}(t_1) \,.\, G_{m_2,k}(t_2) \cdots G_{m_\lambda,k}(t_\lambda) \,.\, dt_\lambda \cdots dt_2 dt_1,$$

is valid and converges uniformly, when $x_1, x_2, \cdots, x_\lambda$ are, respectively, in the circles $|x_1| = r_1$, $|x_2| = r_2, \cdots, |x_\lambda| = r_\lambda$, where

$$r_1 < R_1, \quad r_2 < R_2, \cdots, r_\lambda < R_\lambda,$$

provided k is not equal to a negative integer.

It is evident that corresponding theorems for the expansion of functions of several variables in multiple series of each of the two classes of generalized hypergeometric functions can also be stated, merely replacing the functions $\tilde{F}_{k+m_i}(x_i)$, $G_{m_i,k}(t_i)$ in turn by the functions $\tilde{\mathfrak{F}}_{k+m_i}(x_i)$, $\mathfrak{G}_{m_i,k}(t_i)$ and $\tilde{\boldsymbol{\mathfrak{F}}}_{k+m_i}(x_i)$, $\boldsymbol{\mathfrak{G}}_{m_i,k}(t_i)$, and noting that for the second class the circles of convergence may have any finite radii.

UNIVERSITY OF ILLINOIS,
April, 1924.

Some Properties of the Exponential Mean.

BY JULIA DALE.

Sec. N
Book N
LI

1. INTRODUCTION.

Among the many definitions that have been given for evaluating a divergent series, the exponential mean is frequently one of the most convenient. In this paper, the purpose has been to study some of the properties of this definition and show its relation to Euler's Summation Process.

Unless otherwise stated, we use the symbol

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots,$$

and define

$$x_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

The exponential mean[*] assigns to a divergent series the value $\lim\limits_{n\to\infty} y_n$, if this limit exists, where

$$(E_r) \qquad y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \, \frac{(r-1)^{n-k}}{r^{n-1}} \, x_k.$$

The successive coefficients are the terms of the binomial expansion of

$$[(1-1/r) + 1/r]^{n-1}.$$

The transformation which we have designated in this paper as Euler's Summation Process was defined by Euler,[†] when applied to a series of decreasing terms of alternating signs, in the following manner:

$$S = s_{n-1} + (-1)^{n-1}[\tfrac{1}{2}u_n + \tfrac{1}{4}\Delta^1 u_n + \tfrac{1}{8}\Delta^2 u_n + \cdots + \tfrac{1}{2}{}^k \Delta^{k-1} u_n].$$

where n is an integer and

$$s_{n-1} = u_1 - u_2 + u_3 + \cdots + (-1)^{n-1} u_{n-1}$$

and

$$\Delta^1 u_n = u_n - u_{n+1},$$

$$\Delta^2 u_n = \Delta^1 u_n - \Delta^1 u_{n+1} = u_n - 2u_{n+1} + u_{n+2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

---

[*] Hurwitz: "Report on topics in the theory of divergent series," *Bulletin of the American Mathematical Society,* Vol. 28 (1922), p. 17. This paper will be referred to as *Report.*

Also, Hausdorff: "Summationsmethoden und Momentfolgen," *Mathematische Zeitschrift,* Vol. 9 (1921), p. 86.

[†] Euler: *Institutiones Calculi differentialis,* Part II, De transformatione serierum, p. 232.

Let us use this method to evaluate the

$$1-\tfrac{1}{2}+\tfrac{1}{3}-1$$

The sum of the first eight terms to six transformation to the next seven terms,

$$\begin{aligned} S &= .634524+\tfrac{1}{2}(.111111)+\tfrac{1}{4}(\\ &\quad 1/16(.000505)+1/32(.0001 \\ &= .634524+.058624 \\ &= .693148. \end{aligned}$$

The actual sum of the series is log 2 =

"To reach this degree of accuracy thousand terms of the original series."

The first discussion of the accura Poncelet.[†] He applies this transform alternating signs and computes a formul with the *n*th term. Poncelet, also, defi formation. It is designated in this pap Process. Kummer,[‡] Markoff,[§] and C forms of this transformation, principa has computed a formula for the limit Euler's Summation Process is applied types. He has also suggested that, i certain types of divergent series, it rend

In this paper we have applied this tr in the generalized form to divergent ser In parts 2 and 3 it is proved that not Process but also the generalized one is

---

[*] Bromwich: *Theory of Infinite Series*, (

[†] Poncelet: "Application de la méthod calcul numérique et à la détermination des *Mathematik von Crelle*, Vol. 13 (1833), p. 1.

[‡] Kummer: "Eine neue Methode, die N ender Reihen zu berchnen," *Journal für Math*

[§] Markoff: *Differenzenrechnung*, Leipzig,

[‖] Catalan: "Mémoire sur la Transform *l'Académie de Belgique*, Vol. 33 (1867), p. 2

[¶] Ames: "Evaluation of Slowly Converg 2, Vol. 3 (1902), p. 185.

[**] The terminology here used is that of

A definition is said to be *regular,* if it evaluates every convergent sequence, giving it the value to which it converges. Also, it is proved that this definition is equivalent [*] to the exponential mean. Two definitions are defined as being *equivalent* when each evaluates to the value l every sequence evaluated by the other to the value l.

Part 4 is devoted to the application of the exponential mean to the Cauchy product of two series. Knopp [‡] built a definition by applying E_2 successively p times to the given series. He investigated some of the properties of this definition, among them the value assigned, if one exists, to the Cauchy product of two series. He proved some theorems that are generalizations of theorems of Abel and Mertens for convergent series. Here we give an explicit definition of summability which evaluates the Cauchy product of two series summable respectively E_r and E_s.

2. THE EULER SUMMATION PROCESS.

We consider the Euler process as applied directly from the first term, that is with $n = 1$ in the formula given in the Introduction.

For the series

$$u_1 - u_2 + u_3 - u_4 + \cdots,$$

define

$$s_n = \sum_{k=1}^{n} \tfrac{1}{2}^k \Delta^{k-1} u_1, \tag{1}$$

where

$$\begin{aligned}
\Delta^0 u_1 &= u_1,\\
\Delta^1 u_1 &= u_1 - u_2,\\
\Delta^2 u_1 &= \Delta^1 u_1 - \Delta^1 u_2 = u_1 - 2u_2 + u_3,\\
&\cdots\cdots\cdots\\
\Delta^k u_1 &= \Delta^{k-1} u_1 - \Delta^{k-1} u_2\\
&= u_1 - k u_2 + \frac{k(k-1)}{1\cdot 2} u_3 - \cdots + (-1)^k u_{k+1},\\
&\cdots\cdots\cdots\cdots\cdots;
\end{aligned}$$

then the generalized value of the series is to be

$$s = \lim_{n\to\infty} s_n.$$

---

[*] The terminology here used is that of Professor Hurwitz, *Report.*

[‡] Knopp: "Über das Eulersche Summierungsverfahren," *Mathematische Zeitschrift,* Vol. 15 (1922-23), p. 226; Vol. 18 (1923), p. 125.

In order that the definition may be applied to a series written in tl usual way with positive signs, we define

$$v_n = (-1)^{n-1} u_n.$$

We have

$$s_n = \sum_{k=1}^{n} \tfrac{1}{2}^k \left[u_1 - (k-1) u_2 + \frac{(k-1)(k-2)}{1 \cdot 2} u_3 - \cdots \cdots + (-1)^{k-1} u_k \right],$$

or, by inversion of the order of summation,

$$s_n = \sum_{k=1}^{n} \sum_{h=1}^{n-k+1} \frac{(-1)^{k-1}}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!} u_k. \tag{2}$$

Thus the relation of s_n to v_n is given by

$$s_n = \sum_{k=1}^{n} a_{n,k} v_k, \tag{$\mathcal{E}_2$}$$

where

$$a_{n,k} = \sum_{h=1}^{n-k+1} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}. \tag{3}$$

THEOREM I. *$\mathcal{E}_2$ is a regular transformation.*

In order that a transformation be regular, when applied directly to th terms of the series, it is necessary and sufficient that *

a) for each k, $\lim_{n \to \infty} a_{n,k} = 1$

b) for all n $\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} |$ is bounded.

Obviously, condition a) is satisfied, since

$$\begin{aligned} \lim_{n \to \infty} a_{n,k} &= \lim_{n \to \infty} \sum_{h=1}^{n-k+1} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!} \\ &= \sum_{h=1}^{\infty} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!} \\ &= 1/2^k \left[1 - \tfrac{1}{2}\right]^{-k} = 1. \end{aligned}$$

In order to establish condition b), we write 3) in the form

$$a_{n,k} = 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}.$$

---

* Carmichael: "General Aspects of the Theory of Summable Series," *Bulletin oj the American Mathematical Society,* Vol. 25 (1916-1919), p. 97.

Then

4) $$a_{n,k} - a_{n,k+1} = 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}$$

$$- \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{(h-1)!(k-1)!} \left[\frac{1}{h} - \frac{1}{k}\right]$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{(h-1)!(k-1)!} \left[\frac{1}{h} - \frac{1}{k} + \frac{2}{k}\right]$$

$$- \sum_{h=1}^{n-k} \frac{1}{2^{h+k-1}} \frac{(h+k-1)!}{(h-1)!\,k!}$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k)!}{h!\,k!}$$

$$- \sum_{h=0}^{n-k-1} \frac{1}{2^{h+k}} \frac{(h+k)!}{h!\,k!}$$

$$= \frac{1}{2^n} \frac{n!}{(n-k)!\,k!} .$$

From this, we see that

$$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} | = \frac{1}{2^n} \left[(1+1)^n - 2\right] = 1 - \frac{1}{2^{n-1}} < 1$$

Then

$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} |$ is bounded for all n and $\mathscr{E}_2$ satisfies the conditions for regularity.

We shall now express this transformation in the form of a sequence-to-sequence transformation. Form the sequence

5) $$x_1, x_2, x_3, x_4, \cdots$$

where $x_k = v_1 + v_2 + \cdots + v_k$.

then $v_k = x_k - x_{k-1}$

and

($\mathscr{E}_2$) $$s_n = \sum_{k=1}^{n} a_{n,k}(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} a_{n,k}x_k - \sum_{k=0}^{n-1} a_{n,k+1}x_k$$

$$= \sum_{k=1}^{n-1} (a_{n,k} - a_{n,k+1})x_k + a_{n,n}x_n.$$

Thus

$$s_n = \sum_{k=1}^{n} b_{n,k} x_k,$$

where

$$b_{n,k} = \begin{cases} a_{n,k} - a_{n,k+1} \\ a_{n,n}. \end{cases}$$

Furthermore, by 4), we find

$$b_{n,k} = \frac{1}{2^n} \frac{n!}{(n-k)!\,k!}$$

THEOREM II. *$\mathcal{E}_2$ is equivalent t*

In this work the notation $A \sim B$ sequences A and B are equivalent.

Define

$$\text{(P)} \qquad x'_k = \begin{cases} x_{k-1} & k \\ 0 & k \end{cases}$$

It is readily verified that

$$\text{6)} \qquad E_2 P = P \mathcal{E}_2.$$

The transformation $\mathcal{E}_2$ has been obvious that

$$\text{7)} \qquad \mathcal{E}_2 \sim P \mathcal{E}_2.$$

If a sequence is summable E_r, ther element is summable to the same value

$$\text{8)} \qquad E_2 \sim E_2 P.$$

By means of 6), 7), 8), we see th

$$E_2 \sim \mathcal{E}_2.$$

3. THE GENERALIZED EU

For the generalized Euler Summ

$$u_1 t - u_2 t^2 + u_3 t^3 - u_4 t^4 +$$

we define

$$\text{9)} \qquad s_n = \sum_{k=1}^{n} (t/1+t)^k \Delta^{k-1} u_1,$$

---

* Hurwitz, *Report.*

vhere

$$\Delta^0 u_1 = u_1,$$
$$\Delta^1 u_1 = u_1 - u_2,$$
$$\Delta^2 u_1 = \Delta^1 u_1 - \Delta^1 u_2 = u_1 - 2u_2 + u_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\Delta^k u_1 = \Delta^{k-1} u_1 - \Delta^{k-1} u_2 = u_1 - k u_2 + \frac{k(k-1)}{1 \cdot 2} u_3 - \cdots$$
$$\cdots + (-1)^k u_{k+1}$$

hen the generalized value of the series is to be given by

$$s = \lim_{n \to \infty} s_n.$$

In order to find an expression for s_n, involving directly the terms of th eries, we define

$$v_n = (-1)^{n-1} u_n t^n.$$

We have

$$s_n = \sum_{k=1}^{n} (t/1+t)^k \left[u - \frac{(k-1)}{1!} u_2 + \frac{(k-1)(k-2)}{1 \cdot 2} u_3 \cdots + (-1)^{k-1} u_k\right].$$

›r, by inversion of the order of summation,

10) $$s_n = \sum_{k=1}^{n} \sum_{h=1}^{n-k+1} \frac{(-1)^{k+1}(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}} u_k t^k.$$

Chus the relation of s_n to v_n is given by

$(\mathcal{E}_{1+t})$ $$s_n = \sum_{k=1}^{n} a_{n,k} v_k,$$

vhere

11) $$a_{n,k} = \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}.$$

THEOREM III. *$\mathcal{E}_{1+t}$, where $t > 0$, is a regular transformation.*

Returning to the conditions for regularity as stated in part 2, we fin ondition a) satisfied, since

$$\lim_{n \to \infty} a_{n,k} = \lim_{n \to \infty} \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}$$
$$= \sum_{h=1}^{\infty} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}$$
$$= \frac{1}{(1+t)^k} \left[1 - \frac{t}{1+t}\right]^{-k} = 1.$$

In order to establish condition b) we

$$12)\qquad a_{n,k} - a_{n,k+1} = \sum_{h=1}^{n-k+1} \frac{(h + k -}{(h - 1)!(}$$

$$- \sum_{h=1}^{n-k} \frac{(h + k}{(h -}$$

Changing the variable of summation

$$a_{n,k} - a_{n,k+1} = \sum_{h=0}^{n-k} \frac{(h + k - 1}{h!(k - 1)}$$

$$- \sum_{h=1}^{n-k} \frac{(h + k}{(h - 1}$$

$$= \frac{1}{(1 + t)^k} + \sum_{h=1}^{n-k} \frac{(h +}{h}$$

$$= \frac{1}{(1 + t)^k} + \sum_{h=1}^{n-k} \frac{(h + k - 1)!}{h!\,k!}$$

$$- \sum_{h=1}^{n-k} \frac{(h + k - 1)!}{h!\,k!} \quad \frac{t}{(1 +}$$

$$= \frac{1}{(1 + t)^k} + \sum_{h=1}^{n-k} \frac{(h + k)!}{h!\,k!} \quad \frac{t^h}{(1 + t)^{h+k}}$$

Changing the variable of summation

$$a_{n,k} - a_{n,k+1} = \frac{1}{(1 + t)^k} + \sum_{h=1}^{n-k}$$

$$= \frac{t^{n-k}}{(1 + t)^n} \quad \frac{n}{(n -}$$

Hence, we see that

$$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} | = (t/1 + t)^n [(\,$$

$$= 1 - (t/1 + t$$

Consequently, condition b) is satisfied

Let us express the generalized Euler

---

* We may choose to consider $\mathcal{E}_{1+t}$ with c
of t, with the convention that $\mathcal{E}_1$ is the identit
be extended to show that $\mathcal{E}_{1+t}$ is regular if an
more, for these values $\mathcal{E}_{1+t}$ is totally regular
Report, p. 29.

form of a sequence-to-sequence transformation. As in the preceding case, we find,

$$s_n = \sum_{k=1}^{n-1} (a_{n,k} - a_{n,k+1}) x_k + a_{n,n} x_n,$$

Thus

$$s_n = \sum_{k=1}^{n} \beta_{n,k} x_k,$$

where

$$\beta_{n,k} = \frac{t^{n-k}}{(1+t)^n} \cdot \frac{n!}{(n-k)!\,k!}$$

THEOREM IV. *$\mathcal{E}_{1+t}$, for $t > 0$, is equivalent to the exponential mean E_{1+t}.*

As before we can show that

13) $$E_{1+t}P = P\,\mathcal{E}_{1+t}.$$

The transformation $\mathcal{E}_{1+t}$ has been shown to be regular, then it is obvious that

14) $$\mathcal{E}_{1+t} \sim P\,\mathcal{E}_{1+t}$$

as also [*]

15) $$E_{1+t} \sim E_{1+t}P.$$

It follows from 13), 14), 15) that

$$E_{1+t} \sim \mathcal{E}_{1+t}.$$

THEOREM V. *$\mathcal{E}_{1+t}\,\mathcal{E}_{1+s} = \mathcal{E}_{(1+t)(1+s)}$, for $t > 0$ and $s > 0$.*

Let us apply $\mathcal{E}_{(1+t)(1+s)}$ to 5), then we have

16) $$w_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} \, \frac{(t+s+ts)^{n-k}}{(1+t)^n (1+s)^n} \, x_k.$$

Applying $\mathcal{E}_{1+t}$ to sequence 5) and then $\mathcal{E}_{1+s}$, we find

$$y_n = \sum_{h=1}^{n} \frac{n!}{(n-h)!\,h} \, \frac{t^{n-h}}{(1+t)^n} \sum_{k=1}^{h} \frac{h!}{(h-k)!\,k!} \, \frac{s^{h-k}}{(1+s)^h} \, x_k.$$

$$= \sum_{k=1}^{n} (t/1+t)^n x_k \sum_{h=k}^{n} \frac{n!}{(n-h)!\,(h-k)!\,k!} \, \frac{1}{t^h} \, \frac{s^{h-k}}{(1+s)^h}.$$

$$= \sum_{k=1}^{n} (t/1+t)^n \, \frac{n!}{(n-k)!\,k!} \, \frac{1}{t^k} \, \frac{1}{(1+s)^k} \left[1 + \frac{s}{t(1+s)}\right]^{n-k} x_k.$$

---

[*] **Hurwitz: *Report*, p. 31.**

that is

17) $$y_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$$

Therefore from 16) and 17

By a successive application

COROLLARY 1. $\mathcal{E}_{(1+t_1)(1}$

COROLLARY 2. $\mathcal{E}^{p}_{1+t} =$

THEOREM VI. *If a se*
sequence obtained by omitti
mable $\mathcal{E}_{1+t}$ to the value l.

If a sequence is summal
by omitting or prefixing a
proved Theorem IV, that

By means of the definition

It is sometimes conveni
of the series

$$u_1 t$$

and then apply the generali
terms. The generalized sum

$$\lim_{n\to\infty} s_n$$

where

$$s_n = s_p + \sum_{k=1}^{n} (t,$$

and

$$s_p = u_1 t - u_2 t^2$$

But here again, in orde
the terms of the series, we m

$$v_n = (-1)^{n-1} u_n$$

Then we have

($\mathcal{E}_{1+t,p}$) $$s_n = x_p + \sum_{k=1}^{n} a_n$$

where $a_{n,k}$ is given by 11).

---

* Hurwitz: *Report*, p. 31.

EOREM VII. *$\mathcal{E}_{1+t,p}$ is equivalent to $\mathcal{E}_{1+t}$.*

can obviously write

$$_{t,p}(x_n) = x_p + \mathcal{E}_{1+t}(x_{p+n} - x_p)$$

$$= [x_p - \mathcal{E}_{1+t}(x_p)] + \mathcal{E}_{1+t}(x_{p+n}).$$

ression in the brackets has the limit zero as $n \to \infty$ since $\mathcal{E}_{1+t}$ is r
ius if $\mathcal{E}_{1+t,p}$ evaluates (x_n) to l, $\mathcal{E}_{1+t}$ will evaluate (x_{p+n}) to l,
rem VI, $\mathcal{E}_{1+t}$ will evaluate (x_n) to l. Conversely, if $\mathcal{E}_{1+t}$ evalu
l, $\mathcal{E}_{1+t}$ will evaluate (x_{p+n}) to l and $\mathcal{E}_{1+t,p}$ will evaluate (x_n) 1
re may state

$$\mathcal{E}_{1+t,p} \sim \mathcal{E}_{1+t}.$$

4. MULTIPLICATION OF SERIES.

e expressions obtained by multiplying the terms of one series $\sum u$
is of another series $\sum u_n$ may be arranged formally as a double s
We may collect the terms of this double series into a simple se
means of the Cauchy product, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

ore taking up the case of the summability of the Cauchy produc
es, which are summable E_r, we prove two lemmas.
ce two sequences

$$(x_n) \ : \ x_1, x_2, x_3, \cdots,$$
$$(y_n) \ : \ y_1, y_2, y_3, \cdots,$$

infinite matrix

$$,k) \ : \ \left\| \begin{array}{cccccccccc} a_{11}, & 0, & 0, & 0, & . & . & . & . & . & . \\ a_{21}, & a_{22}, & 0, & 0, & . & . & . & . & . & . \\ a_{31}, & a_{32}, & a_{33}, & 0, & 0, & . & . & . & . & . \\ . & . & . & & & & & & & \\ . & . & . & & & & & & & \\ . & . & . & & & & & & & \\ . & . & . & & & & & & & \end{array} \right\| .$$

en the transformation

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k y_{n-k+1}$$

defines the sequence

$$(z_n) \ : \ z_1, z_2, z_3, \cdots .$$

We indicate the transformation 18) by $\mathfrak{A}$, and the relation by

$$z_n = \mathfrak{A}(x_n y_n).$$

LEMMA 1. *The necessary and sufficient conditions that, if* $\lim\limits_{n\to\infty}$ *and* $\lim\limits_{n\to\infty} x_n = x$, *then* $\lim\limits_{n\to\infty} z_n = xy$, *where*

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k y_{n-k+1}$$

are

a) $\lim\limits_{n\to\infty} a_{n,k} = 0$ *for each k;*

b) $\lim\limits_{n\to\infty} a_{n,n-k+1} = 0,$ *for each k;*

c) $\sum\limits_{k=1}^{n} | a_{n,k} | < c,$ *for all n;*

d) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_{n,k} = 1.$

We shall first establish the necessity of these conditions.

Take

$$y_1 = y_2 = y_n = \cdots = 1$$

and leave (x_n) arbitrary. Substiuting this value of (y_n) in 18), w

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k.$$

This is now an ordinary sequence-to-sequence transformation. transformation of this type to be regular, it is necessary and sufficient

a′) $\lim\limits_{n\to\infty} a_{n,k} = 0,$ for each k

b′) $\sum\limits_{k=1}^{n} | a_{n,k} | < c,$ for all n,

c′) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_{n,k} = 1.$

Hence, we have established the necessity of three of these conditions.

Take

$$x_1 = x_2 = x_3 = \cdots = 1$$

---

* Hurwitz: *Report*, p. 19.

and keep (y_n) arbitrary. Substituting in 18) we have

$$z_n = \sum_{k=1}^{n} a_{n,k} y_{n-k+1}$$

$$= \sum_{k=1}^{n} a_{n,n-k+1} y_k.$$

This is a regular transformation; hence in addition to b′) and c′)

$$\lim_{n\to\infty} a_{n,n-k+1} = 0, \qquad \text{for each } k.$$

It remains to establish the sufficiency of these conditions. Write

19) $$z_n = \sum_{k=1}^{n} b_{n,k} x_k,$$

where

$$b_{n,k} = a_{n,k} y_{n-k+1}$$

From condition a) we see that

20) $$\lim_{n\to\infty} b_{n,k} = 0, \qquad \text{for a fixed } k.$$

By condition c) and the fact that, for all n, $|y_n| \leq Y$, we have

21) $$\sum_{k=1}^{n} |b_{n,k}| = \sum_{k=1}^{n} |a_{n,k}| \; |y_{n-k+1}| < Y \cdot c.$$

We now write

$$\sum_{k=1}^{n} b_{n,k} = \sum_{k=1}^{n} a_{n,k} y_{n-k+1}$$

$$= \sum_{k=1}^{n} a_{n,n-k+1} y_k$$

Since the transformation having the matrix $(a_{n,n-k+1})$ is a regular transformation, we have

22) $$\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k} = y$$

Choose p, such that given $\epsilon > 0$,

$$|x_k - x| < \epsilon/2Yc, \qquad k > p.$$

From 19) we may write

$$z_n = \sum_{k=1}^{p} b_{n,k} x_k + \sum_{k=p+1}^{n} b_{n,k} x_k,$$

23) $$\left| z_n - x \sum_{k=1}^{n} b_{n,k} \right| \leq \sum_{k=1}^{p} |b_{n,k}| \; |x_k - x| + \sum_{k=p+1}^{n} |b_{n,k}| \; |x_k - x|$$

For the fixed number p

$$| x_k - x | < M,$$

Substituting these results in 23),

$$| z_n - x \sum_{k=1}^{n} b_{n,k} | \leq M$$

By 20), given $\epsilon > 0$, we may cho

$$| b_{n,k} | < \epsilon/2pM,$$

Thus we have, for $n > N$,

$$| z_n - x \sum_{k=1}^{n} b_{n,k} | \leq \epsilon/$$

By 22) we see that

$$\lim_{n\to\infty} z_n = xy.$$

The matrix $\mathcal{A}$, which we hav
of two sequences, may be conside
formation of a single sequence

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k;$$

we may then speak of the algebra
for transformations of this form.

A transformation of type $\mathcal{A}$
provided the matrix of $\mathcal{A}$ is pern

LEMMA 2. *A necessary and*

$$\mathcal{A}[E_r(x_n),\ E_r(y_n)] =$$

is that the transformation $\mathcal{A}$ *be*

To show that this condition

$$y_1 = y_2 = y_3 = \cdots$$

thus we have

24) $$\mathcal{A}[E_r(x_n), 1] =$$

Define

25) $$\mathcal{A}(x_n, 1) = A(x_n),$$

ere $A(x_n)$ is an ordinary sequence-to-sequence transformation.

Write Euler's Transformation [*]

)
$$y_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}(n-1)!}{(n-k)!(k-1)!} x_k,$$

ich we shall call Δ. It satisfies the condition

$$\Delta^2 = I,$$

ere I is the identity, as is evident by an actual repetition of 26).

Form the transformation $\Delta E_r \Delta$ and designate it by E'_r, similarly $M' =$ $I\Delta$. The transformation E_r is permutable [†] with M, consequently

$$E'_r M' = M' E'_r$$

necessary and sufficient condition that a transformation be permutable wit ' is that it be a multiplication, i. e. of the form [‡]

$$y_n = f_n x_n.$$

nce, E'_r is a multiplication and may be expressed in the form

$$y_n = \frac{1}{r^{n-1}} x_n.$$

Form the transformation $A' = \Delta A \Delta$, and denote the elements of tl trix of A' by $(a'_{n,k})$. Applying $A'E'_r$ to (x_n), we have

)
$$\omega_n = \sum_{k=1}^{n} a'_{n,k} \frac{1}{r^{k-1}} x_k,$$

plying $E'_r A'$ to (x_n), we obtain

)
$$\Omega_n = \sum_{k=1}^{n} \frac{1}{r^{n-1}} a'_{n,k} x_k.$$

In order that $\omega_n = \Omega_n$, for all (x), we see from 27) and 28) that a'_n 0, when $k \neq n$.

Hence, A' is a multiplication and we have at once that A is permutab th M. From 25) we see that the matrix of $\mathcal{A}$ is the same as the matr A, then $\mathcal{A}$ is permutable with M.

It now remains to prove the sufficiency of this condition. All tran formations permutable with M may be exp

---

[*] Bromwich: *An Introduction to the Theory of Infinite Series* (1908), p. 303.

[†] Hurwitz and Silverman: "On the Consistency and Equivalence of Certain De ions of Summability," *Transactions of American Mathematical Society*, Vol. 1 6. This paper will be referred to as Hurwitz and Silverman.

[‡] Hurwitz and Silverman, p. 6.

$$y_n = \sum_{k=1}^{n} \sum_{h=k}^{n} \frac{(-1)^{h-k}(n-}{(n-h)!(h-k)}$$

$\mathfrak{a}$ is in this class, so we may write

$$a_{n,k} = \sum_{h=k}^{n} \frac{(-1)^{h-k}(n-1}{(n-h)!(h-k)!(}$$

In 18) replace (x_n) by $E_r(x_n)$ and (

$$Z_n = \mathfrak{a}\,[E_r(x_n),\ E_r(y_n)]$$

we have

$$Z_n = \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)}$$

$$\times \sum_{q=1}^{h-k+1} \frac{(h-k)!}{(h-k-q+1)!(q-1)}$$

29) $$Z_n = \sum_{h=1}^{n} \sum_{l=1}^{h} \sum_{q=1}^{h-l+1} \sum_{k=l}^{h-q+1} \frac{}{(n-h)!(k-l}$$

We find

$$\sum_{k=l}^{h-q+1} \frac{(-1)^k}{(k-l)!(h-k-q+1)!} =$$

Substituting this result in 29), we obtain

30) $$Z_n = \sum_{h=1}^{n} \sum_{l=1}^{h} \frac{(-1)^{h-l}(n-1)}{(n-h)!(l-1)!(h-}$$

Applying E_r to (z_n) and writing

$$W_n = E_r\,\mathfrak{a}\,(x_n,\ y_n),$$

we have

$$W_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \quad \frac{(r-}{r^n}$$

$$\times \sum_{h=1}^{k} \sum_{l=1}^{h} \frac{(k-1)!(-1)^h}{(k-h)!(h-l)!(l-}$$

31) $$W_n = \sum_{l=1}^{n} \sum_{h=l}^{n} \sum_{k=h}^{n} \frac{(n-1)!(-}{(n-k)!(k-h)!(h}$$

we find that

$$\sum_{k=h}^{n} \frac{1}{(n-k)!(k-h)!} \cdot \frac{1}{(r-1)^k} = \frac{}{(n-}$$

$$=$$

---

* Hurwitz and Silverman, p. 7.

When we substitute this result in 31), we see that

$$32) \qquad W_n = \sum_{l=1}^{n} \sum_{h=l}^{n} \frac{(n-1)!(-1)^{h-l}}{(n-h)!(h-l)!(l-1)!} f_h \frac{1}{r^{h-1}} x_l y_{h-l+1};$$

By comparing results of 30) and 32), we find that

$$W_n = Z_n,$$

or

$$\mathcal{A}[E_r(x_n), E_r(y_n)] = E_r \mathcal{A}(x_n y_n)$$

as we wished to prove.

Take the two sequences (x_n) and (y_n) and the infinite matrix

$$(m_{nk}) : \left\| \begin{array}{ccccccc} 1/1, & 0, & 0, & 0, & . & . & . \\ 1/2, & 1/2, & 0, & 0, & . & . & . \\ 1/3, & 1/3, & 1/3, & 0, & 0, & . & . \\ . & . & . & . & . & . & . \\ 1/n, & 1/n, & 1/n, & \ldots, & 1/n, & 0 & . \\ . & . & . & . & . & . & . \end{array} \right\|$$

Then the transformation

$$33) \qquad z_n = \sum_{k=1}^{n} m_{nk} x_k y_{n-k+1}$$

defines a sequence (z_n). We indicate the transformation by $\mathcal{M}$, and the relation by

$$(z_n) = \mathcal{M}(x_n y_n).$$

THEOREM VIII. *If $\sum u_n$ is summable E_r to the value σ and $\sum u_n$ is summable E_r to the value τ, then the Cauchy product $\sum w_n$ is summable ME_r to the value $\sigma\tau$.*

The two sequences formed from the two series are, respectively:

$$U_n = u_1 + u_2 + \cdots + u_n,$$
$$V_n = v_1 + v_2 + \cdots + v_n.$$

We shall now form the Cauchy product of the two series, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

The sequence $W_1, W_2, \cdots$, is formed where

$$\begin{aligned} W_n &= w_1 + w_2 + \cdots + w_n \\ &= u_1 V_n + u_2 V_{n-1} + \cdots + u_n V_1, \end{aligned}$$

and

34) $$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n} U_i V_{n-i+1}.$$

Applying M and then E_r to (W_n), we have

35) $$y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \frac{1}{k} \sum_{h=1}^{k} W_k.$$

We shall now apply $\mathscr{M}$ to (U_n) and (V_n), and by 33) we obtain

$$z_n = 1/n \sum_{k=1}^{n} U_k V_{n-k+1}.$$

Applying E_r to (z_n), we have

36) $$z_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \frac{1}{k} \sum_{h=1}^{k} U_h W_{k-h+1},$$

or

37) $$Z_n = E_r \mathscr{M}(U_n, V_n).$$

From 34), 35), 36), we see that

$$y_n = Z_n.$$

If now we show that $\lim_{n\to\infty} Z_n$ exists and equals $\sigma\tau$, our theorem is proved.

By Lemma 2, we have

38) $$E_r \mathscr{M}(U_n V_n) = \mathscr{M}[E_r(U_n), E_r(V_n)].$$

Let

$$\sigma_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} U_k,$$

$$\tau_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} V_k.$$

As $\sum u_n$, $\sum v_n$ are summable E_r,

$$\lim_{n\to\infty} \sigma_n = \sigma, \qquad \lim_{n\to\infty} \tau_n = \tau.$$

We have

39) $$\sigma_k = \sum_{p=1}^{k} \frac{(k-1)!}{(k-p)!(p-1)!} \frac{(r-1)^{k-p}}{r^{k-1}} U_p$$

similarly

40) $$\tau_{n-k+1} = \sum_{p=1}^{n-k+1} \frac{(n-k)!}{(n-k-p+1)!(p-1)!} \frac{(r-1)^{n-p-k+1}}{r^{n-p}} V_p.$$

From 37) and 38) we have

$$Z_n = \mathscr{M}[E_r(U_n), E_r(V_n)],$$

then substituting the values as given by 39) and 40) we find

$$Z_n = 1/n \sum_{k=1}^{n} \sigma_k \tau_{n-k+1}.$$

Obviously the conditions of Lemma 1 are satisfied and we state that

$$\lim_{n\to\infty} Z_n = \sigma\tau.$$

COROLLARY 1. *If* $\sum u_n$ *is summable* E_r *to the value* σ *and* $\sum v_n$ *is summable* E_s *to the value* τ*, then if* $r > s$*, the Cauchy product* $\sum w_n$ *is summable* ME_r *to the value* $\sigma\tau$.

This follows at once since the exponential mean becomes more inclusive as r increases.[*]

COROLLARY 2. *If* $\sum u_n$*,* $\sum v_n$*, and the Cauchy product* $\sum w_n$ *are summable*[†] E_r *to the values* σ*,* τ*,* ω *respectively, then* $\omega = \sigma\tau$.

If C_p denotes the Cesaro mean of order p then[†]

41) $$C_{\mu+\nu+1}(W_n) = \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

where $\mathfrak{D}$ is the transformation

$$Z_n = \sum_{k=1}^{n} d_{n,k} x_k y_k,$$

with

$$d_{n,k} = \frac{(\mu+\nu+1)!(n-1)!(n+k-1)!(n-k+\nu)!}{\mu!\nu!(n-k)!(k-1)!(\mu+\nu+n)!}.$$

This transformation satisfies the conditions of Lemma 1; this consideration furnishes the usual proof[‡] of the theorem: *If* $\sum u_n$ *is summable* C_μ *to the value* σ *and* $\sum v_n$ *is summable to value* τ*, then the Cauchy product* $\sum w_n$ *is summable* $C_{\mu+\nu+1}$ *to value* $\sigma\tau$.

We proceed to generalize this result.

---

[*] Hurwitz: *Report*, p. 27.

[†] Knopp: "Über das Eulersche Summierungsverfahren," *Mathematische Zeitschrift*, Vol. 15 (1922-23), p. 226.

Knopp gives a definition formed by applying E_2 successively p times. He proves for this definition the results stated in this corollary. But since $(E_2)^p = E_{2p}$, his result is the special case in which r, instead of being any number greater than 1, is restricted to be an even integer.

[‡] Chapman: "Non-Integral Orders of Summability of Series and Integrals," *Proceedings of London Mathematical Society*, Series 2, Vol. 9 (1910), p. 369.

THEOREM IX. *If $\sum u_n$ is summable $C_\mu E_r$ to the value σ and summable $C_\nu E_r$ to the value τ, then their Cauchy product is su $C_{\mu+\nu+1}E_r$ to the value $\sigma\tau$.*

Since the transformation $\mathfrak{D}$ is permutable with M, we have

$$42)\qquad \mathfrak{D}[C_\mu E_r(U_n), C_\nu E_r(V_n)] = E_r\,\mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

multiplying each side of 41) on the left by E_r, we obtain

$$43)\qquad E_r C_{\mu+\nu+1}(W_n) = E_r\,\mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

From 42) and 43) we have

$$44)\qquad E_r C_{\mu+\nu+1}(W_n) = \mathfrak{D}[C_\mu E_r(U_n), C_\nu E_r(V_n)].$$

Thus we see from 44) that (W_n) is summable $E_r C_{\mu+\nu+1}$ to the value σ

COROLLARY 1. *If $\sum u_n$ is summable $C_\mu E_r$ to the value σ, and summable $C_\nu E_s$ to the value τ, then if $r > s$, the Cauchy product is su $C_{\mu+\nu+1}E_r$ to the value $\sigma\tau$.*

CORNELL UNIVERSITY,
June, 1924.

The Eliminant of a Net of Curves.*

BY FRANK MORLEY.

A process was given by Sylvester † for writing in determinant form the condition that three homogeneous functions of three variables, of the same order, have a common zero, or that three curves of the same order, in a plane, have a common point.

The process is evidently not in final form, and Clebsch ‡ remarks "mais le caractère invariant n'en ressort pas clairement."

A memoir of A. L. Dixon § should also be mentioned, but as the method gives more than enough equations it is not applicable here. If instead of considering the discriminant as an invariant under the group of projections, we were considering it under the group of inversions, this memoir would be highly pertinent.

In what follows I obtain the eliminant as a determinant analogous to Sylvester's but obviously invariantive. Further, this determinant, properly bordered, is shown to be that form which gives the common point when it exists.

§ 1. *Salmon's connex.*

Salmon,‖ considering a plane curve and its tangent at a point x, wrote a connex which cuts out the remaining intersections of the tangent and the curve. It is of the form

$$A^3x^{2(m-1)}y^{m-1}$$

when the curve is Ax^{m+1}. A^3 indicates that the coefficients are of the third degree in the coefficients A.

The connex naturally occurs in the theory of Abelian integrals of the third kind.¶

If now we remark that when the curve has a double point x, y becomes arbitrary, we have at once from Salmon's connex $m(m+1)/2$ curves of order $2m-2$ on the double point.

As all first polars are also on the double point, we have 3 curves of order m which give $3(m-1)m/2$ independent curves of order $2m-2$, so that in all we have a basis of $m(2m-1)$ curves of order $2m-2$, whence the dis-

---

* Read at the International Congress of Mathematicians, Toronto, 12 Aug., 1924.

† Salmon, *Higher Algebra*, § 91; Sylvester, *Works*, v. 1, p. 75 or *Phil. Mag.*, 1841.

‡ Clebsch-Lindemann. *Lectures*, v. 2. p. 13 of the French translation.

§ London Math. Soc., s. 2, v. 7, p. 49.

‖ Salmon, *Higher Plane Curves*, § 393 or *Phil. Mag.*, 1858; Cayley, *Works*, v. 4; Bramble, *Johns Hopkins Circulars*, July, 1915.

¶ Klein, "Abelsche Funktionen," *Math. Ann.*, v. 36 or *Works*, v. 3, p. 408.

criminant is written down as a determinant.
adapt Salmon's connex to a net which is not

§ 2. *The linear syzygy.*

Consider the sum of homogeneous produ

$$H^n \equiv \sum a^n + \sum a^{n-1}b + \sum a^{n-2}b$$

and let this, when a is 0, become $H_a{}^n$.
Then evidently $H^n - aH^{n-1} = H$

Multiply the 4 such equations by numbers a_1,

2.1) $\sum a_1 . H^n - \sum aa_1 . H^{n-1} =$

Let now

$$\begin{aligned} a_1 &= (ax) \mid \beta\gamma\delta \mid \\ b_1 &= -(\beta x) \mid \gamma\delta a \mid \\ c_1 &= (\gamma x) \mid \delta a\beta \mid \\ d_1 &= -(\delta x) \mid a\beta\gamma \mid \end{aligned}$$

and

where x and y are points of a plane.

Then $\sum a_1 = 0$ and

Hence from 1) $\sum a_1$

that is $\sum (ax)^{n+1} \mid \beta\gamma\delta \mid$

where $h_a{}^n$ is the sum of homogeneous produ
$(\beta x)(\gamma y)(\delta x)$, $(\beta x)(\gamma x)(\delta y)$. If then w
$(\gamma x)^m$, $(\delta x)^m$ there is the syzygy, linear in t

$$\sum (ax)^m \mid \beta\gamma\delta \mid h_a{}^{m-1} = 0,$$

2.2) $\sum^4 (\delta x)^m \mid a\beta\gamma \mid \sum (ax)^{\mu+\nu} (\beta x)^{\nu+\lambda} ($

where $\lambda + \mu + \nu = m - 1$.

The same process gives a linear syzygy fo

§ 3. *The eliminant.*

Hence if the curves a, β, γ have a co
is not 0,

3.1) $\mid a\beta\gamma \mid h^{m-1} =$

where we have written h for h_δ.

This is the required connex. If we de
Bx^m, Cx^m the connex may be denoted by

$$ABCx^{2(m-1)}y^m$$

When x is a common point, y is arbitrary. Thus we have $m(m+1)/2$ curves of order $2(m-1)$ on the common point, and the 3 given m-ics contribute $3(m-1)m/2$ curves of order $2(m-1)$, so that we have a basis of $m(2m-1)$ curves from which we eliminate.

The resulting determinant may be indicated by

3.2).
$$\Delta \equiv \begin{array}{r|c|} (m-1)m/2 & A \\ \text{``} & B \\ \text{``} & C \\ m(m+1)/2 & ABC \end{array}$$

§ 4. *A second form of the eliminant.*

The connex 3.1) is a combination of polars of y as to the Jacobian of the net, and as to the Jacobians of polars, as in Salmon's case (*Higher Plane Curves*, § 394). Other linear connexes which vanish at a common point are

4.1)
$$(\alpha\beta\gamma)(\alpha x)(\beta x)(\gamma x)h^{m-2}$$

or
$$ABCx^{2m-1}y^{m-2},$$

and generally
$$(\alpha\beta\gamma)(\alpha x)^r(\beta x)^r(\gamma x)^r h^{m-1-r}$$

or
$$ABCx^{2-m2+r}y^{m-1-r}.$$

For there is a syzygy
$$\sum(\alpha x)^m(\beta\gamma\delta)(\beta x)^r(\gamma x)^r(\delta x)^r h_{\alpha}{}^{m-1-r} = 0.$$

The series of connexes terminates with the Jacobian J, when $r = m-1$. When $r = m-2$ the connex is the polar of y as to J.

The form 4.1) also gives a basis. For it gives $(m-1)m/2$ curves of order $2m-1$, and the 3 given curves give $3m(m+1)/2$ curves of order $2m-1$, whence we have, for a common point, the eliminant as a determinant of $m(2m+1)$ rows,

$$\Delta' \equiv \begin{array}{r|c|} m(m+1)/2 & A \\ \text{``} & B \\ \text{``} & C \\ (m-1)m/2 & ABC \end{array}$$

§ 5. *The apolar forms.*

When we know that three curves have a common point, the next question is to find that point.

Let $(\alpha x)^m$ be a point-form and $(a\xi)^n$ a line-form and let $n >$ or $= m$. The polar of $(\alpha x)^m$ as to $(a\xi)^m$ is by definition

$$(\alpha a)^m (a\xi)^{n-m}.$$

When this vanishes, for an arbitrary line ξ, the forms are apolar (or incident). Apolarity means then $\binom{n-m+2}{2}$ conditions linear in either set of coefficients. Thus 3 conics have 3 independent apolar conics and a unique apolar cubic. Three cubics have 7 apolar cubics, 6 apolar quartics, 3 apolar quintics, and a unique apolar sextic. In the latter case if the cubics be $(\alpha x)^3$, $(\beta x)^3$, $(\gamma x)^3$ and the sextic $(s\xi)^6$, there are 30 equations $(\alpha a)^3(a\xi)^3 = 0$, but among them are three identities such as

$$(\alpha s)^3(\beta s)^3 = (\beta s)^3(\alpha s)^3.$$

To find this sextic, we may ask more generally for a sextic $(s\xi)^6$ such that the polars of $(\alpha x)^3$, $(\beta x)^3$, $(\gamma x)^3$ as to it be given cubics, that is

$$\begin{gathered}(\alpha s)^3(s\xi)^3 \equiv \lambda(a\xi)^3\\ (\beta s)^3(s\xi)^3 \equiv \mu(b\xi)^3\\ (\gamma s)^3(s\xi)^3 \equiv \nu(c\xi)^3\\ (s\xi)^6 = 0.\end{gathered}$$

We have then 31 equations from which to eliminate the 28 coefficients and λ, μ, ν. The resulting determinant has two factors,

$$(a\beta)^3(b\gamma)^3(c\alpha)^3 - (a\gamma)^3(b\alpha)^3(c\beta)^3$$

which vanishes if the problem is possible, and a sextic in ξ which is the apolar sextic.

Three quartics $(\alpha x)^4 \cdots$ have a unique apolar ninic $(s\xi)^9$, for among the 3×21 equations are 9 identities

$$(\alpha s)^4(\beta s)^4(s\xi) \equiv (\beta s)^4(\alpha s)^4(s\xi)$$

And in general 3 forms $(\alpha x)^m \cdots$ have a unique apolar form $(s\xi)^{3m-3}$, since

$$1 - 3\binom{m-1}{2} + 3\binom{2m-1}{2} - \binom{3m-1}{2} = 0.$$

Equally for ternary forms of different orders $(\alpha x)^l$, $(\beta x)^m$, $(\gamma x)^n$ there is a unique apolar form $(s\xi)^{l+m+n-3}$ since

$$1 - \Sigma\binom{l-1}{2} + \Sigma\binom{m+n-1}{2} - \binom{l+m+n-1}{2} = 0.$$

And so in general in a flat F_d any $d+1$ forms have a unique apolar form of order $\Sigma m - d - 1$.

A point, on a curve $(ax)^m$, taken m or more times, is apolar to that curve. Thus when 3 curves have a common point c, $(c\xi)^{l+m+n-3}$ is the apolar form. But in this case $(c\xi)$ to any higher power is equally an apolar form. Thus taking the case of a net, there is in general no form of order $> 3m-3$ apolar to the net, unless a single condition is satisfied; this condition is in all cases the vanishing of the eliminant.

§ 6. *The bordered determinant.*

For clearness take a particular net, say a net of cubics. We have 3 cubics Ax^3, Bx^3, Cx^3 and the connex $ABCx^4y^2$ of § 3, which we will write $(\delta x)^4(\epsilon y)^2$. We have a unique apolar sextic, 3 independent apolar quintics, which are polars of points as to the sextic, and 6 independent apolar quartics which are polars of conics as to the sextic.

Consider these quartics $(s\xi)^4$. To name them we use the connex $(\delta x)^4(\epsilon y)^2$. The polar of $(s\xi)^4$ as to this quartic in x is

$$(\delta s)^4(\epsilon y)^2$$

a conic in y which we may call the conic associate with $(s\xi)^4$, and denote by $(\eta y)^2$. Thus an apolar quartic is defined by

$$(s\xi)^4=0, \quad (\alpha s)^3(s\xi)\equiv 0, \quad (\beta s)^3(\beta\xi)\equiv 0, \quad (\gamma s)^3(\gamma\xi)\equiv 0$$

$$\text{and} \quad (\delta s)^4(\epsilon y)^2\equiv\rho(\eta y)^2.$$

Eliminating the 15 coefficients of the quartic and ρ from these 16 equations we have the bordered eliminant

$$\Delta(\xi^4,\eta^2) = \begin{array}{r|cc|} & \multicolumn{2}{c}{15+1} \\ \hline 3 & A & 0 \\ 3 & B & 0 \\ 3 & C & 0 \\ 6 & ABC & \eta^2 \\ 1 & \xi^4 & 0 \end{array}$$

This for given η is an apolar quartic. For given ξ it is the conic apolar to all associate conics of quartics on ξ.

If now $\Delta=0$, then $\rho=0$. The determinant factors into $(c\xi)^4$, where c is the common point, and a conic in η which is apolar to the associate conics of the apolar quartics, of which now that $(c\xi)^4$ is singled out there are only ∞^5.

§ 7. *The conic factor.*

When $\Delta=0$ there is a common point c. It is to be shown that the cor factor of $\Delta(\xi^4 y^2)$ is then $(cy)^2$.

The 15 quartics have now an apolar quartic. There is thus an ident among them. That is $Ax^3 . A_1 x + Bx^3 . B_1 x + Cx^3 . Cx \equiv ABCx^4 y^2$.

The fundamental syzygy (2.2) with $m=3$ and δ a line on y, so th $(\delta y)=0$, becomes

$$7.1) \qquad |\alpha\beta\gamma|\, h^2(\delta x)^3 \equiv \sum(\alpha x)^3\, |\beta\gamma\delta|\, (\delta x)^2 \\ \{(\beta y)^2(\gamma x)^2 + (\beta y)(\gamma y)(\beta x)(\gamma x) + (\beta x)^2(\gamma y)^2\}.$$

Let y be 1, 0, 0 and let this be a common point so that $\alpha_0{}^3 = \beta_0{}^3 = \gamma_0{}^3$ $\delta_0 = 0$. Then

$$7.2) \qquad |\alpha\beta\gamma|\, h^2(\delta x) \equiv \sum(\alpha x)^3 . |\beta\gamma\delta| \\ \{\beta_0{}^2(\gamma x)^2 + \beta_0\gamma_0(\beta x)(\gamma x) + \gamma_0{}^2(\beta x)^2\}.$$

The cofactor of $(\alpha x)^3$ contains the factor (δx). For writing $\delta_2 =$ $\delta_1 = -x_2$ so that $|\beta\gamma\delta|$ is $\beta_0(\gamma x) - \gamma_0(\beta x)$, it is

$$\beta_0{}^3(\gamma x)^3 - \gamma_0{}^3(\beta x)^3$$

that is, 0. Hence (δx) divides each individual term in 7.2).

Thus the sought identity is

$$7.3) \qquad ABCx^4c^2 \equiv \sum Ax^3 . A_1 x$$

and therefore when $\Delta = 0$ all apolar quartics $(s\xi)^4$ are apolar also to the fo

$$ABCx^4c^2.$$

The associate conics are therefore all on the point c. Thus the conic fac is simply $(c\eta)^2$.

§ 8. *The unique apolar form.*

Thus when $\Delta=0$, $\Delta(\xi^4, \eta^2) = (c\xi)^4(c\eta)^2$, and in particular $\Delta(\xi^4, \xi^2)$ $(c\xi)^6$. This is what the unique apolar sextic becomes when $\Delta = 0$, and sin this unique sextic is of lower degree in the coefficients than Δ we infer th *the apolar sextic is in general*

$$\Delta(\xi^4, \xi^2).$$

Similarly if we border the second form of the eliminant we obta $\Delta(\xi^5, \eta)$ which represents all apolar quintics. When $\Delta = 0$, the form comes $(c\xi)^5(c\eta)$, and $\Delta(\xi^5, \xi)$ in another form of the apolar sextic, in p ticular when $\Delta = 0$, and therefore in general. So generally, by parallel argument, for a net of curves of order m, $\Delta(\xi^{2m-2}, \eta^{m-1})$ gives the apolar curves of order $2m-2$, $\Delta'(\xi^{2m-1}, \eta^{m})$ gives the apolar curves of order $2m-1$, and when $\eta \equiv \xi$ either form becomes the unique apolar curve of order $3m-3$.

§ 9. *The case of several common points.*

The identical vanishing of the bordered eliminant means that the net has 2 common points. This case can be handled by bordering the eliminant twice. Thus for 3 conics we consider

$$\Delta(\xi^2\xi'^2, \eta\eta') \equiv \begin{array}{c|ccc|} & 6 & 1 & 1 \\ \hline 1 & A & 0 & 0 \\ 1 & B & 0 & 0 \\ 1 & C & 0 & 0 \\ 3 & ABC & \eta & \eta' \\ 1 & \xi^2 & 0 & 0 \\ 1 & \xi'^2 & 0 & 0 \end{array}$$

When the net has two common points this equation in $\eta\eta'$, replacing $\overline{\eta\eta'}$ by x, in the join of the common points. And in general for n common points of any net we are to consider a determinant form of the eliminant, bordered n times.

On an Equation of Planar Motion.

By Frank Morley.

---

§ 1. *The equation.*

The points of a plane being named by complex numbers, let a point of the base circle, whose centre is 0 and radius 1, be denoted by t.

Then the general rotation, or displacement of a plane (say horizontal) over another, is given by

1.1) $$y = tx - a$$

where t and a are given.

We regard this not as an alibi but as an alias, y and x being the names in the two planes of two points in the same vertical.

From the ∞^3 equations (1) we now select ∞^1 by writing

$$a = f(t).$$

We have then for varying t an equation of motion,

1.2) $$y = tx - f(t).$$

§ 2. *Pins and slots.*

If in equation 2) we fix x, we have an equation which maps the base circle onto the points of the y-plane which successively coincide with x; that is, the path traced on the y-plane by a pin fixed in the x-plane.

Similarly if we fix a pin in the y-plane we have in (2) the map-equation of the path traced in the x-plane.

If we make slots in the planes along two such paths, we have two pins moving in two slots, and the motion is thus mechanically defined. The two pins may be in the one plane, and therefore the two slots in the other, or we may have a pin and a slot in each plane.

§ 3. *The cusp-loci, or centrodes.*

A curve expressed as a map of the base circle,

$$x = \phi(t),$$

has a cusp when there is a value of t for which

$$dx/dt = 0.$$

Thus the paths in the y-plane, for which x is given, have cusps when $dy/dt = 0$, or

3.1) $$x = f'(t).$$

and those in the x-plane, for which y is given, have cusps when $dx/dt = 0$, or

3.2) $$y = tf'(t) - f(t).$$

These are the centrodes, and the motion is the rolling of the one on the other. But this is, mechanically, no simplification. For instance the rolling of a circle on a circle is effected by the sliding motion of cycloidal teeth.

It is to be noted that when the curvatures of the centrodes at the common point are equal, all paths have a cusp. Thus cusps are not always on the centrodes.

§ 4. *Mechanical description of polynomic curves.*

We suppose the curve given as a map of the base-circle,

$$y = \phi(t).$$

If in $\phi(t)$ occurs a term ct^n, we replace the constant c by a variable x, so that

$$y = xt^n + \phi_1(t),$$

an equation of the form (1.2).

The slots in which move the pins $x = 0$, $y = 0$ are then

$$y = \phi_1(t), \qquad x = -\phi_1(t)/t^n.$$

These slots are in general simpler than the original curve; and to their description the same method may be applied.

We thus get rid of the terms in $\phi(t)$ which are of the form ct^n, and if $\phi(t)$ consists of a finite number of such terms the method will be practical.

The standard case is when y is a polynomial in t, say

$$y + a_1 t + a_2 t^2 + \cdots + a_n t^n = 0.$$

We replace this by the equation of motion

$$y_0 + a_1 t + a_2 t^2 + \cdots + x t^n = 0,$$

and the slots are

$$y + a_1 t + \cdots + a_{n-1} t^{n-1} = 0$$

and $$x + a_1/t^{n-1} + \cdots + a_{n-1}/t = 0$$

where $$y = xt^n.$$

§ 5. *Mechanical description of rational curves.*

If the curve be given by

$$5.1) \qquad y = \sum_1^n \frac{A_k}{a_k - t},$$

we write $(a_n - t)(\bar{a}_n - 1/\tau) = a_n \bar{a}_n - 1$.

Expressed by proper fractions in τ, y becomes

$$y = B_0 + B_1/\tau + \sum_1^{n-1} \frac{B_k}{b_k - \tau}.$$

Moving the base-point to B_0, and replacing B_1 by x, we have the equa motion in the form

$$y = x/\tau + \sum_1^{n-1} \frac{B_k}{b_k - \tau}.$$

When $x = 0$, the slot is

$$y = \sum_1^{n-1} \frac{B_k}{b_k - \tau}$$

and when $y = 0$, the slot is

$$x = \sum_1^{n-1} \frac{B_k}{1 - b_k/\tau}.$$

Repeating the process for each slot, we come to the case of two circula or 3-bar motion (rational).

There remains the more general case when in the given expressic in partial fractions there are repeated poles, as for instance a term $A/(a$ Geometrically speaking, the curve has then a focus at infinity. The ea is to allow inversions. After a suitable inversion, any rational curve pressed in the form (5.1), the foci being now all finite.

Thus the general rational curve, after an inversion when necessa be mechanically described by two pins moving in two slots, the slots simpler than the proposed curve.

An Extension of the Problem of the Elastic Bar.*

BY H. T. DAVIS.

1. *The Problem of the Elastic Bar.* It is well known † that the problem of determining the transverse vibrations of an elastic bar leads to the study of an ordinary linear differential equation of fourth order with four linear boundary conditions, two of these conditions holding at one end of the bar and two at the other.

For example, if $w = f(x)$ is the initial elastic curve of a bar with fixed end points at $x = a$ and $x = b$, which starts vibrating without initial velocity, the ordinate of a point on the elastic curve satisfies the following partial differential equation:

$$\frac{\partial^4 W}{\partial x^4} + \phi(x) \frac{\partial^2 W}{\partial t^2} = 0,$$

and the following boundary conditions,

$$W(a, t) = W(b, t) = 0,$$
$$W'_x(a, t) = W'_x(b, t) = 0,$$

where $\phi(x)$ is a function of positive sign depending upon the physical properties of the bar.

If we assume that the solution is of the form

$$W(x, t) = u(x) \cos\sqrt{\lambda}\, t,$$

we are led to a study of the ordinary equation

$$\frac{d^4 u}{dx^4} - \lambda\phi(x) u = 0,$$

and the boundary conditions,

$$u(a) = u(b) = 0,$$
$$u'(a) = u'(b) = 0.$$

The problem of determining *characteristic functions* for this system and the expansion of $f(x)$ in terms of these functions is the problem of the elastic

---

* Presented to the American Mathematical Society, Dec. 29, 1922 and Dec. 26, 1924.

† See Lord Rayleigh's *Theory of Sound*, Vol. 1, chap. 8.

bar, which has been extensively studied
memoir considered asymptotic developı
his dissertation in 1906 computed G
boundary conditions in the homogeneou
W. Stekloff and J. Tamarkine ‡ have
homogeneous case, and in 1913 C. I
results for special cases of the probler
in the application of the theory of integ
of which is given in *Die Integralgleich*

Now if we set

$$L(u) \equiv \frac{d^2}{dx^2} (p u'')$$

it is clear that the problem presented
following differential system:

$$L(u) + \lambda k$$

$$U_i(u) \equiv \sum_{k=1}^{4} a_{ik} u^{(k-1)}$$

$$U_j(u) \equiv \sum_{k=1}^{4} b_{jk} u^{(k-1)}$$

where $p(x) \neq 0$, $k(x) \neq 0$ in the inte

In this paper we propose to develo
acteristic numbers and characteristic
down under the additional assumption
adjoint.¶ A method due to G. D. Birk
asymptotic form of the characteristic
paper consists in associating with syst

---

* *Annales de l'école normale supérieure*
See in particular page 405.

† Dissertation, Göttingen (1906).

‡ *Rendiconti di Palermo*, Vol. 31 (191
Rendus, Vol. 154 (1912), pp. 269-271.

§ *Phil. Mag.* (6), Vol. 25 (1913), pp. 8

‖ Second edition (Braunschweig), (1922

¶ The idea of adjoint equations and ac
been largely developed by Birkhoff and Bô
Leçons sur les méthodes de Sturm, Paris,
this paper.

** *Transactions of the Amer. Math. Soc* which connects solutions of system (1) with solutions of the system adjoint to (1). Similar matrices exist for any problem where $L(u)$ is a self-adjoint expression of even order and much of the work that follows can be generalized for equations of higher order.

Boundary conditions in which $U_i(u)$ is a linear combination of $u, \cdots, u^{(n-1)}$ at one point only will be called *Sturmian boundary conditions,* since it was C. Sturm who in 1836 first studied such conditions associated with a homogeneous differential equation of second order.* It is important to notice that in the case of equations of order greater than three, different types of Sturmian boundary conditions appear. For example, in the case $n=4$, we may have three conditions at one point and one at the other or, as in the case of the vibrating bar, we may have two conditions at each point. The problems presented by these two cases are very different from one another, for it is not difficult to show that *the boundary conditions in the first case can never be self-adjoint, while in the second case only two conditions on the coefficients are necessary for self-adjointness.*†

2. *A Preliminary Normalization.* We begin by considering the well-known Lagrange identity

$$\int_a^b [v\,L(u) - u\,L(v)]\,dx = P(u,v), \qquad (2)$$

---

* *Journal de Mathématiques,* Vol. 1 (1836), pp. 106-186.

† This fact can be proved by means of a theorem due to D. Jackson. *Transactions of the Amer. Math. Soc.,* Vol. 17 (1916), pp. 418-424. Thus suppose that

$$\Pi = \begin{pmatrix} \pi_1 & \pi_3 \\ \pi_4 & \pi_2 \end{pmatrix}$$

is the matrix of the bilinear form $P(u,v)$ in equation (2), section 2. Let A_1 be the square matrix of the coefficients of n variables in $U_i(u)$, these coefficients being so chosen that A_1 is non-singular. Then A_2 will be the square matrix of the remaining n variables. If δ denotes the product $A_1^{-1}A_2$ and δ' the conjugate of δ then Jackson's theorem asserts:

"If the differential expression $L(u)$ is self-adjoint, the condition that the boundary conditions be self-adjoint is that the matrix

$$\delta'\Pi_1\delta - 2\Pi_4\delta + \Pi_2$$

be symmetric; if $L(u)$ is an anti-self-adjoint expression or differs from such an expression only in the term of order zero, the condition is that the matrix just written down be skew-symmetric."

The proof of the fact in question consists in showing that for any choice of a non-singular matrix A_1, we are led to the contradiction that the determinant of A_1 must be zero. The author hopes to be able to state later the corresponding facts for Sturmian systems of any order.

in which $P(u, v)$ is a non-singular bilinear form in the eight quantities $u(a), \cdots, u'''(a); u(b), \cdots, u'''(b);\ v(a), \cdots, v'''(a); v(b), \cdots, v'''(b)$.

For the purpose of simplifying the coefficients of system (1) and reducing it to a normal form, we make in (2) the following transformation of both dependent and independent variables:

$$\begin{aligned} u(x) &= \phi(x) y(x), \qquad x = \xi(t). \\ v(x) &= \phi(x) w(x), \end{aligned}$$

Indicating by a bar the value of a function of x after its transformation into a function of t, we shall have

$$\int_a^b [v\, L(u) - u\, L(v)]\, dx =$$

$$\int_{t_0}^{t_1} \bar{\phi}^2 \bar{p} \left(\frac{d\xi}{dt}\right)^{-3} \bar{w} \left\{ \frac{d^4 \bar{y}}{dt^4} + \cdots + [\cdots + \lambda \bar{k} \left(\frac{d\xi}{dt}\right)^4] \bar{y} \right\} dt - \qquad (3)$$

$$\int_{t_0}^{t_1} \bar{\phi}^2 \bar{p} \left(\frac{d\xi}{dt}\right)^{-3} \bar{y} \left\{ \frac{d^4 \bar{w}}{dt^4} + \cdots + [\cdots + \lambda \bar{k} \left(\frac{d\xi}{dt}\right)^4] \bar{w} \right\} dt = P^*(\bar{y}, \bar{w}),$$

where P^* is the bilinear form in terms of the new variables $\bar{y}$ and $\bar{w}$.

We now notice that *the property of self-adjointness will be preserved providing the Lagrange identity remains invariant under the transformation.*

Therefore if we set,

$$\bar{\phi}^2 \bar{p} \left(\frac{d\xi}{dt}\right)^{-3} = 1, \qquad \left(\frac{d\xi}{dt}\right)^4 \bar{k} = \pm 1, \qquad (4)$$

where the ambiguous sign is determined from the sign of $\bar{k}$ so as to make the transformation real, it is clear that the resulting differential expression $L^*(y)$ will be self-adjoint and of the form

$$L^*(y) = \frac{d^4 y}{dt^4} + \frac{d^2}{dt^2}\ (Q\, y') + R\, y + \lambda\, y. \qquad (5)$$

We must next see that the boundary conditions of system (1) have also remained self-adjoint under the transformation defined by (4).

Suppose that we employ the notation

$$U_i(u) = a_{i1} u_1 + \cdots + a_{i8} u_8 \quad (i = 1, \cdots, 4), \qquad (6)$$

$$V_j(u) = \beta_{j1} u_1 + \cdots + \beta_{j8} u_8 \quad (j = 5, \cdots, 8), \qquad (7)$$

where u_1 to u_8 are the eight values $u(a),\ u'(a),\ \cdots,\ u'''(a); u(b), u'(b),$

$\cdots, u'''(b)$ and where U_i and V_j are connected by the identity*

$$P(u, v) \equiv \sum_{i,j=1}^{8} U_i V_j,$$

in which the first n V's and the last n U's are arbitrarily chosen.

From the definition of adjoint boundary conditions there must exist constants c_{jk} such that

$$V_j = c_{j1} U_1 + c_{j2} U_2 + \cdots + c_{j4} U_4, \quad (j = 5, \cdots, 8). \tag{8}$$

Upon substituting (6) and (7) in (8) and equating the coefficients of like terms we obtain four sets of eight equations of the form

$$c_{11}\alpha_{1i} + c_{12}\alpha_{2i} + \cdots + c_{14}\alpha_{4i} = \beta_{ji}, \qquad (i = 1, \cdots, 8;\ j = 5, \cdots, 8). \tag{9}$$

If A denotes the matrix of this system of equations and B the augmented matrix.

$$A = \begin{pmatrix} \alpha_{11}, & \cdots, & \alpha_{41} \\ \cdot & \cdot & \cdot \\ \alpha_{18}, & \cdots, & \alpha_{48} \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_{11}, & \cdots, & \alpha_{41}, & -\beta_{j1} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{18}, & \cdots, & \alpha_{48}, & -\beta_{j4} \end{pmatrix},$$

then a necessary and sufficient condition that the equation be consistent is that the ranks of A and B shall be equal.†

Suppose that the rank of A is r. By hypothesis the system is self-adjoint so that the c_{ij} in (9) actually exist. Consequently all of the determinants in B of order higher than r must vanish. If the condition be added that the U_i shall be linearly independent, then *the ranks of A and B must both be equal to n.*

From Leibnitz's formula it follows that (6) and (7) are submitted to the transformation

$$u^{(r)}(a) = \phi^{(r)}(a)y(a) + \tbinom{r}{1}\phi^{(r-1)}(a)y'(a) + \cdots + \tbinom{r}{r}\phi(a)y^{(r)}(a),$$
$$u^{(r)}(b) = \phi^{(r)}(b)y(b) + \tbinom{r}{1}\phi^{(r-1)}(b)y'(b) + \cdots + \tbinom{r}{r}\phi(b)y^{(r)}(b),$$

where $r = 0, \cdots, 3$.

Denote by C the matrix of this transformation. This matrix is clearly non-singular since its determinant reduces to the product of the principal diagonal which equals $[\phi(a)]^4[\phi(b)]^4$. By hypothesis on $p(x)$ and $k(x)$ this is different from zero.

Consequently B^*, the matrix of the transformed coefficients will be the

---

[*] See Bôcher's *Méthodes de Sturm, loc. cit.*, chapter 2.

[†] Bôcher, *Introduction to Higher Algebra*, New York (1922), p. 46.

following product,

$$B^* = C\,B.$$

Since B is of rank n by hypothesis and since the first matrix is no singular, it follows that the rank of B^* must be n also.* A similar stateme applies to $A\dagger = C\,A$.

The transformation of the dependent variable may be similarly treat since the determinant of the matrix of the linear transformation on A^* a B^* defined by it is easily shown to be $\left\{\left[\frac{d\xi(t_0)}{dt}\right]\left[\frac{d\xi(t_0)}{dt}\right]\right\}^{-6}$ which non-singular from the hypothesis on $k(x)$.

These results may be summarized in the following theorem:

Theorem 1. *The self-adjointness property of both the differential equ tion and the boundary conditions of system* (1) *is unaltered by the tra formations defined by equations* (4).

3. *The Fundamental Matrix and Its Properties.* Let us recall † that $(u_1, \cdots, u_n)$ is a fundamental set of solutions of the general equation nth order

$L(u) \equiv p_0(x)u^{(n)}(x) + p_1(x)u^{(n-1)}(x) + \cdots p_n(x)u(x) = 0$, and W, the Wronskian, is the determinant

$$W = \begin{vmatrix} u_1 & u'_1 & u''_1 & \cdots & u_1^{(n-1)} \\ u_2 & u'_2 & u''_2 & \cdots & u_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ u_n & u'_n & u''_n & \cdots & u_n^{(n-1)} \end{vmatrix},$$

then $v_i = \dfrac{1}{p_0}\dfrac{\partial \log W}{\partial u_i^{(n-1)}}$, $i = 1, \cdots, n$, will be a set of linearly independe solutions of the adjoint equation. These solutions are called by Darboux t *adjoints* of the solutions u_i and satisfy the relations

$$\begin{aligned} v_1u_1^{(j)} + v_2u_2^{(j)} + \cdots + v_nu_n^{(j)} &= 0, \qquad j < n-1, \\ v_1u^{(n-1)} + v_2u^{(n-1)} + \cdots + v_nu^{(n-1)} &= 1/p_0. \end{aligned} \qquad (1$$

In case, however, that the equation is self-adjoint (or anti-self-adjoin it is clear that the v_i must be linear combinations of $(u_1, \cdots, u_n)$,

---

* Bôcher, *Higher Algebra*, p. 79, theorem 7.

† For an extensive development of this subject see G. Darboux, *La théorie Surfaces*, Paris, (1889), part 2, book 4, chap. 5; also L. Schlesinger, *Handbuch Theorie der linearen Differentialgleichungen*, Leipzig, (1895), Vol. 1, chap. 3.

$$v_i = c_{i1}u_1 + c_{i2}u_2 + \cdots + c_{in}u_n, \quad (i = 1, \cdots, n). \tag{11}$$

The matrix of this transformation we shall designate by C, and we shall show that it has the following fundamental property:

Theorem 2. If $L(u) = 0$ is a self-adjoint (or anti-self-adjoint) equation of order n and if $(v_1, \cdots, v_n)$ are the adjoints of the solutions $(u_1, \cdots, u_n)$, then the matrix C of the substitution (11) *is either symmetric or skew-symmetric as n is odd or even.*

To prove this we first obtain from the n equations (11), n sets of n equations by differentiating v_i, $n - 1$ times and from these we can solve for the values of the c_{ij} in terms of v_i and u_j and their $n - 1$ derivatives. Thus, explicitly, we have

$$c_{ij} = [D_0 v_i + D_1 v'_i + \cdots + D_{n-1} v_i^{(n-1)}] \; 1/W, \quad (i = 1, \cdots, n), \tag{12}$$

where D_k is the cofactor of $v_i^{(k)}$ in the determinant which we obtain by replacing the jth column of W by $v_i, v'_i, \cdots, v_i^{(n-1)}$.

But from equations (10) we observe that we have

$$D_{n-1} = p_0 v_j W.$$

By successive differentiation of this equation and the use of the differential equation in eliminating terms which involve $u_j^{(n)}$, we obtain the following expansions:

$$\begin{aligned}
D_0 &= (-1)^{n-1}[(p_0 v_j)^{(n-1)} - (p_1 v_j)^{(n-2)} + \cdots + (-1)^{n-1}(p_{n-1} v_j)]\, W, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
D_{n-3} &= [(p_0 v_j)'' - (p_1 v_j)' + (p_2 v_j)]\, W, \\
D_{n-2} &= -[(p_0 v_j)' - (p_1 v_j)]\, W, \\
D_{n-1} &= (p_0 v_j)\, W.
\end{aligned} \tag{13}$$

If we call Π the matrix of this substitution we observe that it is identical with the matrix of the bilinear form $P(u, v)$ in equation (2) which, in the self-adjoint case, is known to be symmetric when n is an odd number and skew-symmetric when n is an even number.*

When this substitution is made in (12), c_{ij} is expressed as a quadratic form in $v_i v_j$ whose matrix is symmetric when n is odd and skew symmetric when n is even. This proves the theorem.

We can obtain now by means of theorem 2 some elegant and important

---

* L. Schlesinger, *loc. cit.*, Vol. 1, pp. 70-71, 73-74.

identities satisfied by the second order m
$n = 4$, where we assume the differential

$$L(u) \equiv u^{(IV)} + p_2 u'' + $$

For convenience we shall designate b

$$\begin{pmatrix} i & j \\ l & m \end{pmatrix} = ($$

these minors of second order which are (
If we substitute the values of v_i (i =
tities (10), and those derived from (10)
time recall theorem 2, we have at once
between the minors:*

$$c_{12}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix} + c_{23}($$

$$c_{12}\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} + c_{23}($$

$$c_{12}\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix} + c_{23}($$

$$c_{12}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} + c_{23}($$

$$c_{12}\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} + c_{23}($$

$$c_{12}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} + c_{13}\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} + c_{14}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + c_{23}($$

That system (14), which may be lo
in the six variables c_{ij}, is consistent for
is readily seen as follows. The sixth o
of all of the minors of the second orde
determinant W_2, then by the theorem of

$$W_2 = W^{\binom{3}{1}} =$$

The explicit determination of the
plished as follows. Suppose that the fu

---

* The linear dependence of the minor
of (14) has long been known. In an article
équations différentielles linéaires du quatri
(1883), p. 329, they are shown to satisfy an
Philosophical Transactions, Vol. 179 (1888),
relations and their connection with matrix C
† G. Kowalewski, *Einführung in die Dete*

been uniquely determined by the following relations at some point $x = a$:

$$a_{i1}u(a) + a_{i2}u'(a) + a_{i3}u''(a) + a_{i4}u'''(a) = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol which is equal to l when $i = j$ and equal to 0 when $i \neq j$, and where the determinant $D = | a_{ij} |$ is different from zero.

Then if A_{ij} is the cofactor of a_{ij} in the determinant D, it follows that

$$u_i^{(j-1)}(a) = A_{ij}/D \text{ and } W = (u_1, u'_2, u''_3, u'''_4) = D^3/D^4 = 1/D. \qquad (15)$$

Moreover, since

$$c_{11}u_1^{(j-1)} + c_{12}u_2^{(j-1)} + c_{13}u_3^{(j-1)} + c_{14}u_4^{(j-1)} = v_1^{(j-1)}, \quad (j = 1, 2, 3, 4),$$

we have

$$c_{1j} = (D_{j0}/W)v_1 + (D_{j1}/W)v_1' + (D_{j2}/W)v_1'' + (D_{j3}/W)v_1''',$$

where D_{jk} is the cofactor of $u_j^{(k)}$ in the determinant W.

But these cofactors are easily computed in terms of the a_{ij} if we make use of well known theorems on adjoint determinants.

Thus

$$D_{01}/W = D(D^2/D^3)a_{11} = a_{11}, \cdots, D_{jk}/W = a_{j,k+1},$$

from which it follows that

$$c_{ij} = a_{j1}v_i + a_{j2}v_i' + a_{j3}v_i'' + a_{j4}v_i'''.$$

Making use of equations (13) we can easily compute v_i. The values thus obtained are tabulated for convenience below:

i	1	2	3	4
$v(a)$	a_{14}	a_{24}	a_{34}	a_{44}
$v'(a)$	$-a_{13}$	$-a_{23}$	$-a_{33}$	$-a_{43}$
$v''(a)$	$a_{12}-p_2(a)a_{14}$	$a_{22}-p_2(a)a_{24}$	$a_{32}-p_2(a)a_{34}$	$a_{42}-p_2(a)a_4$
$v'''(a)$	$-a_{11}+p_2(a)a_{13}$	$-a_{21}+p_2(a)a_{23}$	$-a_{31}+p_2(a)a_{33}$	$-a_{41}+p_2(a)a_4$

4. *General Separation Theorems.* We now proceed to the examination of certain general results which follow from the relations developed in the last section.

Consider the following two normalized Sturmian systems:

$$L(u) + \lambda u \equiv u^{(IV)} + p_2u'' + p_2'u' + (p_3 + \lambda)u = 0,$$

$$\text{(I)} \quad \begin{array}{l} A_1(u) = A_2(u) = 0, \\ B_1(u) = B_2(u) = 0, \end{array} \qquad \text{(II)} \quad \begin{array}{l} A_3(u) = A_4(u) = 0, \\ B_1(u) = B_2(u) = 0, \end{array} \qquad (16)$$

where the $A_i(u)$ are linear condition linear conditions at the point $x = b$.

If we assume that the determina can choose the fundamental set of sol

$$A_{ij}$$

where δ_{ij} is the Kronecker symbol, a systems will appear as roots of

$$\Delta_1 = \begin{vmatrix} B_{13} & B_{14} \\ B_{23} & B_{24} \end{vmatrix}, \text{ an}$$

respectively.

Suppose, further, that in matri and $c_{12} \neq 0$, $c_{34} \neq 0$. These are co but they are consistent and can be sa

It is easily seen that Δ_2 can be ex

$$\Delta_2 = \gamma_{34}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \gamma_{24}\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

$$+ \gamma_{14}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where γ_{ij} is the second order determi jth columns in the following matrix:

$$\begin{pmatrix} b_{11} & b \\ b_{21} & b \end{pmatrix}$$

Now if each column of the iden γ_{ij} and the resulting equations added multiplied by a non-vanishing consta satisfied:

$$\gamma_{23} - \gamma_{14} - p_2$$

This result is stated in the follo that it has no analogue in Sturmian s

THEOREM 3. *The characteristic are identical providing the following*

---

* Kowalewski, *Determinantentheorie,* (

$$\begin{array}{ll} 1) & c_{13} = c_{14} = c_{23} = c_{24} = 0, \\ 2) & |a_{ij}| \neq 0, \\ 3) & \gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0. \end{array}$$

As an example consider the following systems:

$$\text{(I)} \quad \begin{array}{l} u^{(IV)} - \lambda u = 0, \\ u'(0) = u''(0) = 0, \\ u(\pi) = u'(\pi) = 0, \end{array} \qquad \text{(II)} \quad \begin{array}{l} u^{(IV)} - \lambda u = 0, \\ u(0) = u'''(0) = 0, \\ u(\pi) = u'(\pi) = 0. \end{array}$$

The conditions of the theorem are seen to be satisfied. We choose for a fundamental set of solutions the following:

$$u_1 = e^{\rho x}, \quad u_2 = e^{-\rho x}, \quad u_3 = \sin \rho x, \quad u_4 = \cos \rho x, \tag{18}$$

where $\rho = \sqrt[4]{\lambda}$. In terms of these functions we can now easily compute the two determinants Δ_1 and Δ_2, whose zeros are the characteristic numbers for the two systems (I) and (II). These determinants are found to be identical since

$$\Delta_1 = \Delta_2 = 4\lambda \sinh \rho\pi \sin \rho\pi,$$

and the characteristic numbers are $\lambda = \pm 1^4, \pm 2^4, \pm 3^4, \cdots$.

We next derive a separation theorem for the characteristic numbers of the following Sturmian systems:

$$\begin{array}{lll} & u^{(IV)} + p_2 u'' + p_1' u' + (p_3 + \lambda) u = 0, & \\ \text{(I)} \quad A_1(u) = A_2(u) = 0, & & \text{(II)} \quad A_1(u) = A_2(u) = 0, \\ \phantom{\text{(I)}} \quad B_1(u) = B_2(u) = 0, & & \phantom{\text{(II)}} \quad B_3(u) = B_4(u) = 0. \end{array} \tag{19}$$

In order to discover the conditions on the coefficients of the two systems so that the characteristic numbers of one will alternate or coincide with the characteristic numbers of the second, we shall make use of the following theorem recently published by the author: *

THEOREM. *If the existence of a sequence of characteristic numbers for problem* (I) *has been established, then a sequence of characteristic numbers for problem* (II), *alternating or coincident with the characteristic numbers of problem* (I), *exist provided*

$$\int_a^b [\Gamma_1(x, x) - \Gamma_2(x, x)]\,dx \neq 0, \qquad (\lambda_0 \leqq \lambda \leqq \lambda_m) \tag{20}$$

---

* *Bulletin of the Amer. Math. Soc.*, Vol. 28, (1922), pp. 390-394.

where Γ_1 *and* Γ_2 *are the Green's funct*
respectively.

Let us now recall that we can wr
equation and its boundary conditions

$$\Gamma(x, t) = C_1 u_1(x) + \cdot$$

where $g(x, t)$ is a function defined as

$$g(x, t) = \pm \frac{1}{2W} \begin{vmatrix} u_1(x), & u_1 \\ u_2(x), & u_2 \\ \cdot \quad \cdot & \cdot \\ u_n(x), & u_n \end{vmatrix}$$

Here the plus sign is to be used when
W is the Wronskian or determinant
their first $n-1$ derivatives.

We now notice that, by means o
$g(x, t)$ in the form

$$g(x, t) = \pm \tfrac{1}{2}[u_1(x) v_1(t)$$

where the ambiguous sign is to be det

In order to determine the constan
$\Gamma(x, t)$ in the boundary conditions,
that the C_i are expressible as linear fo
of the two Green's functions in the
form in the variables $u_i(x)$, $v_i(x)$, i.

$$\Gamma_1(x, x) - \Gamma_2(x, x) =$$

Now if we denote by F the mat
matrix of the linear substitution (11),
into a quadratic form in $u_i(x)$, $u_j(x)$

$$T =$$

By a fundamental theorem on
$\Gamma_2(x, t)$ both belong to self-adjoint sys

---

* J. Horn, *Theorie der partiellen Diff*

of x and t and consequently their difference is a symmetric function.[*]

It follows at once, then, since T is the matrix of a bilinear form in $u_i(x)$, $u_j(t)$ which is symmetric that T is itself a symmetric matrix. Consequently when we consider the question of whether or not the quadratic form (21) is definite, we need only to examine the signs of the principal minors of the determinant of T and apply to them a well known criterion.[†]

We next proceed to the calculation of the matrix T for the special Sturmian systems (19). Suppose, for simplicity, that matrix C, by the choice of a proper fundamental system, is the following:

$$C = \begin{bmatrix} 0 & 0 & 0 & c_{14} \\ 0 & 0 & c_{23} & 0 \\ 0 & -c_{23} & 0 & 0 \\ -c_{14} & 0 & 0 & 0 \end{bmatrix}.$$

We see explicitly from the values of c_{ij} computed in the preceding section that this imposes upon the coefficients of $A_1(u)$, $A_2(u)$ the single condition $c_{12}=0$.

If for brevity we use the notation

$$T_{ij} = \begin{vmatrix} B_{1i} & B_{1j} \\ B_{2i} & B_{2j} \end{vmatrix} \quad \text{and } S_{ij} = \begin{vmatrix} B_{3i} & B_{3j} \\ B_{4i} & B_{4j} \end{vmatrix},$$

then the matrices of the Green's functions for the two systems (19) will be M_1 and M_2 where

$$M_1 = \begin{bmatrix} 0, & 0, & 0, & c_{14} \\ 0, & 0, & c_{23}, & 0 \\ 0, & c_{23}, & -2c_{23}T_{24}/T_{34}, & -2c_{14}T_{14}/T_{34} \\ c_{14}, & 0, & 2c_{23}T_{34}/T_{34}, & 2c_{14}T_{13}/T_{34} \end{bmatrix} \tag{22}$$

and where M_2 is an identical array except that T_{ij} has been replaced throughout by S_{ij}.

From (22) we are able to write down the conditions for the self-adjointness of the general Sturmian system of fourth order. Since matrix M_1 must be symmetric we shall have

$$c_{14}T_{14} + c_{23}T_{23} = 0,$$

the condition for which is (17). *Hence the general Sturmian system of fourth order in normal form will be self-adjoint providing its coefficients satisfy the conditions:*

---

[*] Bôcher, *Méthodes de Sturm, loc. cit.*, pp. 104-105.

[†] Kowalewski, *Determinantentheorie, loc. cit.*, pp. 220 and 238.

$$\gamma_{23} - \gamma_1$$

Returning to (22) we observ

$$T$$

so that we have explicitly,

$$T = \begin{vmatrix} 0, & 0, & \\ 0, & 0, & \\ 0, & 0, & -2c_{23}T_{24}/T_{34} \\ 0, & 0, & 2c_{23}T_{23}/T_{34} \end{vmatrix}$$

In order to prove the definite
show that the following determin

$$\Delta = \begin{vmatrix} -S_{34}T_{24} + \\ S_{34}T_{23} - \end{vmatrix}$$

$$= -[S^2_{34}(T_{24}T_{13} -$$
$$+ T_{14}S_{23} + T$$

If we now expand the dete
according to the first two rows we

$$|B_{ij}| = |b_{ij}|\ |u_{ij}| = |b$$
$$= T_{12}S_{34} - T_{13}S_{24} +$$

Also, replacing in (24) the la
as before by Laplace's development

$$T_{12}T_{34} - T$$

and similarly that

$$S_{12}S_{34} - S_1$$

When these values are substitu

$$\Delta = -[S^2_{34}T_{12}T_{34} + S_{34}T$$

$$= -|b_{ij}|\,W\,T_{34}S_{34}.$$

Hence $\Delta = 0$, and the form is

$$|$$

These results may be summari

THEOREM 4. *The characteristic numbers of systems (I) and (II),* (19) *ther coincide or alternate with one another provided that their coefficient tisfy the following conditions:*

1) $c_{12} = 0$,
2) $\gamma_{23} - \gamma_{14} - p_2(b)\gamma_{12} = 0$.
3) $|b_{ij}| = 0$.

A simple example will illustrate the theorem. Consider the systems:

$$(\mathrm{I})\quad \begin{aligned} &u^{(IV)} - \lambda u = 0,\\ &u(0) = u'(0) = 0,\\ &u(\pi) = u'(\pi) = 0; \end{aligned} \qquad (\mathrm{II})\quad \begin{aligned} &u^{(IV)} - \lambda u = 0,\\ &u(0) = u'(0) = 0,\\ &u(\pi) = u''(\pi) = 0. \end{aligned}$$

The coefficients are seen to satisfy the conditions of the theorem.

If we choose as a fundamental system, the one given by (18), we can mpute the two determinants Δ_1 and Δ_2, whose zeros are the characteristic mbers of the two systems (I) and (II). These are found to be

$$\Delta_1 = 1 - \cosh \sqrt[4]{\lambda}\pi \cos \sqrt[4]{\lambda}\pi,$$
$$\Delta_2 = \cosh \sqrt[4]{\lambda}\pi \sin \sqrt[4]{\lambda}\pi - \sinh \sqrt[4]{\lambda}\pi \cos \sqrt[4]{\lambda}\pi.$$

From these expressions we see that the characteristic numbers for the o systems will be

$$\lambda_n = [\tfrac{1}{2}(2n-1) + \epsilon_n]^4, \quad \lambda'_n = [\tfrac{1}{4}(4n-3) + \epsilon'_n]^4,$$

ere ϵ_n and ϵ'_n are numbers which become vanishingly small for large values n. The separation of the characteristic numbers by one another is thus plicitly seen for large values of n and may be verified by a graph for small lues of n.

5. *Asymptotic Distribution of the Characteristic Numbers.* We can ke use of theorem 4 to obtain an existence theorem for the general Sturm- system of fourth order. For this purpose consider the following systems:

$$u^{(IV)} + p_2 u'' + p'_2 u' + (p_3 + \lambda)u = 0,$$

$$(\mathrm{I})\quad \begin{aligned} &A_1(u) = A_2(u) = 0,\\ &B_1(u) = B_2(u) = 0, \end{aligned} \qquad (\mathrm{II})\quad \begin{aligned} &A_1(u) = A_2(u) = 0,\\ &u(b) = u'(b) = 0. \end{aligned} \qquad (25)$$

Choose for a fundamental set of so
$x = a$, the following conditions are sati

$$u_i^{(j-1)}(a) =$$

where δ_{ij} is the Kronecker symbol.

Then in system (II), (25), the
characteristic numbers will be zeros of t

$$F(b, \lambda) = E_{12}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} - E_{13}\Bigl(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$$
$$+ E_{23}\Bigl(\begin{smallmatrix} 0 & 1 \\ 2 & 3 \end{smallmatrix}$$

where E_{ij} is the second order determi
and jth columns in the matrix formed
Furthermore the value of $F(x, \lambda)$
the point $x = a$, and by successive diffe
$\left.\frac{d^4F}{dx^4}\right|_{x=a}$ can be computed also. The ex
ant in application and are recorded bel

$F(a) = E_{12}$, $F'(a) = -E_{13}$, F''
$- E_{12}p'_2(a) - 2E_{24}$, $F^{(IV)}(a) = 2E_{12}$
$E_{12} + 2p_2(a)E_{13} - p_2(a)E_{14} + 2E_{34}$.

But it is known that $F(x, \lambda)$ satis
of fifth order:*

$$\frac{d^5F}{dx^5} + 2p_2\,\frac{d^3F}{dx^3} + 3p'_2\,\frac{d^2}{dx}$$
$$+ (p_2''' + p_2p'_2 - 2$$

Consequently, if we apply the resu
(II), (25), we can reduce the existenc
Sturmian system of fourth order to the
order. Thus we have:

THEOREM 5. *Given an integral* $F($
ly defined by the values (27). *If the*
the following conditions:

$$1)\ c_{12} = 0, \quad 2)\ \gamma_{23} - \gamma$$
$$3)\ (b_{13}b_{24} - b_{14}b_{23}) = 0,$$

---

* *Halphen, loc. cit.*, p. 329.

and if $F(b,\lambda)$ *vanishes m times in the* λ*-interval,* $\Lambda_1 < \lambda < \Lambda_2$, *then there exist at least* $m-1$ *characteristic numbers of system* (I) *which either alternate or coincide with the m* λ*-roots of* $F(b,\lambda)$ *in the interval* (Λ_1, Λ_2).

From this theorem the asymptotic distribution of the characteristic numbers for the general Sturmian system is readily deduced. For this purpose we make use of an important theorem due to Birkhoff,* which asserts that for values of ρ in a region S, the asymptotic form of the solutions of the equation

$$\frac{d^n u}{dx^n} + \rho a_{n-1}(x,\rho)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + \rho^n a_0(x,\rho)u = 0,$$

where $a_i(x,\rho) = \sum_{j=0}^{\infty} a_{ij}(x)\rho^{-j}$, are given by

$$u_i^{(j)} = u_i^{(j)}(x,\rho) + e^{\rho\int_a^x \omega_i(t)dt}\,\frac{E_i}{\rho^{m-j}}, \quad \begin{cases} i = 1, 2, \cdots, n, \\ j = 0, 1, 2, \cdots, n-1, \\ m \text{ arbitrary,} \end{cases}$$

in which $u_i(x,\rho) = e^{\rho\int_a^x \omega_i(t)dt}\sum_{j=0}^{m-1}\frac{u_{ij}(x)}{\rho^j}$, $u_{i0}(x) \neq 0$ for $a \leqq x \leqq b$.

The ω_i are roots of the equation

$$\omega^n + a_{n-1\,0}(x)\omega^{n-1} + \cdots + a_{00}(x) = 0,$$

and the E_j are functions of ρ and x bounded for large values of ρ.

We now apply this theorem to equation (28). Two cases are to be considered according to whether we set $\lambda = -\rho^4$ or $\lambda = +\rho^4$.

First Case. If we choose $\lambda = -\rho^4$, then $a_1 = 4 + \frac{f(x)}{\rho^4}$, so that the values ω_i satisfy the equation

$$\omega^5 + 4\omega = 0. \tag{29}$$

A fundamental set of solutions of equation (28) will then be

$$F_i(x) = e^{\omega_i(x-a)\rho}\left[1 + \frac{F_{i1}(x)}{\rho} + \frac{F_{i2}(x)}{\rho^2} + \cdots\right], \quad (i = 1, 2, \cdots, 5) \tag{30}$$

where $\omega_1 = 0$, $\omega_2 = 1+i$, $\omega_3 = 1-i$, $\omega_4 = -1+i$, $\omega_5 = -1-i$.

---

* G. D. Birkhoff, *Transactions of the Amer. Math. Soc.*, Vol. 9, (1908), p. 225.

In terms of these functions $F(b)$ may

$$F(b) = \frac{1}{W(b)} \sum_{i=1}^{5} \left| \begin{array}{l} F_1(a), \cdots, F_{i-1}(a), F(a), \\ F'_1(a), \cdots, F'_{i-1}(a), F'(a \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F_1^{(IV)}(a), \cdots, F_{i-1}^{(IV)}(a), \\ \qquad\qquad\qquad\qquad F_i \end{array} \right.$$

where $W(b)$ is the Wronskian of the funda

From (27) we see that all of the $F^{(j}$ $F^{(IV)}(a)$ which is linear in λ and equal t observe that in (31) for large positive valu those containing $F_2(b)$ and $F_3(b)$, and si ρ the dominant terms will be those contain coefficients of these terms do not vanish. always different from zero can be seen exp of $1/\rho$ and retaining only those coefficie lowest degree in $1/\rho$. The coefficients of totic to

$$\pm \rho^{10} \{2 F(a) D_{4i} + \frac{F'(a) D_{1i}}{\rho} +$$

where the D_{ki} are the cofactors of the ele column of the following determinant:

$$D = \left| \begin{array}{ll} \omega_2 & \omega_3 \\ \omega_2^2 & \omega_3 \\ \cdot & \cdot \\ \omega_2^4 & \omega_3 \end{array} \right.$$

Since none of these cofactors are zero values $F^{(j-1)}(a)$ can be zero, because $F(b$ follows that none of the coefficients of $F_i($

We next examine several special cases. be found by a simple computation and b terms of (31), that the dominant terms will be

$$A\, e^{(b-a)\rho} \cos \rho (b - a),$$

From the ρ-zeros of this function we of the characteristic numbers, thus obtaini

$$\lambda_i \sim -\left\{\frac{2n-1}{2(b-a)}\pi\right\}^4,$$

for n sufficiently large.

In order to obtain the λ_i which correspond to negative values of ρ, we combine the fourth and fifth terms of (31). These will be found asymptotic to $-\dfrac{2n-1}{2(b-a)}\pi$ so that no new values of λ_i are introduced.

In a similar way we examine the case where $F(a)=0$, $F'(a)\neq 0$. The dominant function is then of the form

$$A\, e^{(b-a)\rho} \sin\{\rho(b-a)+\pi/4\}, \quad A\neq 0,$$

from which it follows that

$$\lambda_i \sim -\left\{\frac{4n-1}{4(b-a)}\pi\right\}^4.$$

Similarly, if $F''(a)$ is the first value different from zero, we obtain $\lambda_i \sim -\{n\pi/(b-a)\}^4$. The case where $F'''(a)$ is the first non-vanishing value gives the same asymptotic form of the characteristic numbers as for $F(a)=0$, $F'(a)\neq 0$, and the case where $F^{(IV)}(a)\neq 0$, corresponds to the first case.

Second Case. If we choose $\lambda=\rho^4$, then the values of ω_i are the roots of $\omega^5-4\omega=0$, or $\omega_1=0$, $\omega_2=\sqrt{2}$, $\omega_3=-\sqrt{2}$, $\omega_4=\sqrt{2i}$, $\omega_5=-\sqrt{2i}$.

The same argument applies as in the preceding case except that the dominant terms are now either $e^{\sqrt{2}\rho(b-a)}$ for positive values of ρ or $e^{-\sqrt{2}\rho(b-a)}$ for negative values of ρ. Hence there will not exist characteristic numbers for arge values of ρ and the λ_i-sequence is finite.

These results may be summarized as follows:

THEOREM 6. *If the coefficients of system (I), (25), satisfy the conditions of theorem 5, then there will exist for negative values of λ an infinite sequence of characteristic numbers, separated by, or coincident with, the following values when n is taken sufficiently large:*

(a) $\lambda_i \sim -\left\{\dfrac{2n-1}{2(b-a)}\pi\right\}^4$, *if $F(a)\neq 0$; or if $F^{(IV)}(a)\neq 0$ and all he preceding values zero.*

(b) $\lambda_i \sim -\left\{\dfrac{4n-1}{4(b-a)}\pi\right\}^4$, *if $F(a)=0$, $F'(a)\neq 0$; or if $F'''(a)\neq$ 0 and all preceding values zero.*

(c) $\lambda_i \sim -\left\{\dfrac{n}{b-a}\pi\right\}^4$, *if $F''(a)\neq 0$ and all preceding values zero.*

For positive values of λ, the number of characteristic values
finite.

As an illustration consider the characteristic numbers for the

$$u^{(IV)} - \lambda u = 0,$$
$$u(0) = u'(0) = 0, \quad u(\pi) = u''(\pi) = 0.$$

By a simple computation we have

$$F(0) = F'(0) = F''(0) = F''(0) = 0, \quad F(0) = 2.$$

Hence the characteristic numbers must alternate or coincide
values $+\left\{\frac{2n-1}{2}\right\}^4$, a result consistent with that obtained in th
ample of the preceding section.

INDIANA UNIVERSITY,
BLOOMINGTON, INDIANA.

On the Sylow Subgroups of the Symmetric and Alternating Groups.

BY LOUIS WEISNER.

1. *Introduction.* It is the object of this paper to exhibit a formula for calculating the number of Sylow subgroups of each order in the symmetric and alternating groups of degree n. Let p be a prime factor of $n!$ and let p^m be the highest power of p that divides $n!$; further let n be written in the scale of p in the form

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.$$

The number of Sylow subgroups of order p^m in the symmetric group of degree n will be shown to be $\dfrac{n!}{a_0!\ a_1! \cdots a_k!\ p^m (p-1)^m}$. Incidentally other properties of the Sylow subgroups of the symmetric group will be noted.

The number of Sylow subgroups of order p^m in the alternating group of degree n is evidently the same as the number in the symmetric group if $p > 2$. It will be shown that for $n > 5$, the alternating group contains as many Sylow subgroups of even order as the symmetric group.

2. *An Invariant Abelian Subgroup.* The substitutions

$$s_1 = (1, 2, \cdots, p), \quad s_2 = (p+1, p+2, \cdots, 2p), \cdots$$
$$s_{n_1} = (\overline{n_1 - 1}\ p + 1, \overline{n_1 - 1}\ p + 2, \cdots, n_1 p),$$

where, with the notation of § 1,

$$n_i = a_i p + a_{i+1} p^2 + \cdots + a_k p^{k-i} \qquad (i = 1, 2, \cdots, k),$$

clearly generate an abelian subgroup I of order p^{n_1} and type $(1, 1, \cdots, 1)$. We proceed to prove that *I is invariant under every Sylow subgroup H of G that contains I,* where G denotes the symmetric group of degree n.

A conjugate I' of I under G contains conjugates of $s_1, s_2, \cdots, s_{n_1}$. At least one of these conjugates must affect one of the symbols $1, 2, \cdots, p$. Let $t = (1, i_2, \cdots, i_p)$ be such a substitution of I', and for definiteness suppose t affects the symbol 1. If H contains I and I', (s_1, t) is in H and hence is of order a power of p, say p^a. Now (s_1) is transitive in the symbols

1, 2, · · · , p, and (t) is transitive in the symbols 1, i_2, · · · , i_p. It follo that (s_1, t) is transitive in the symbols which it affects. Hence p^a is divisil by the degree of (s_1, t) which, being less than $2p$, must equal p. Hence and t affect the same p symbols. Since the symmetric group on p lett involves no subgroup of order p^2, (s_1, t) is of order p. Hence s_1 is a pov of t and is contained in I'. Similarly I' contains $s_2, \ldots, s_{n_1}$. Hei $I' = I$.

3. *Number of Conjugates of I.* We proceed to determine the num of conjugates of I under G by calculating the number of ways of select the cyclic subgroups, each involving a single cycle of p letters, of whicl conjugate of I is the direct product. Denoting by ${}_xP_y$ the number of p mutations of x things taken y at a time, it is clear that s_1 may be chosen ${}_nP_p/p$ ways, and hence (s_1) may be chosen in ${}_nP_p/p(p-1)$ ways. Af (s_1) is chosen, (s_2) may be chosen in ${}_{n-p}P_p/p(p-1)$ ways; and then (may be chosen in ${}_{n-2p}P_p/p(p-1)$ ways; etc. It follows that the cy groups, each involving a single cycle of p letters, of which a conjugate o is the direct product, can be chosen in

$$\frac{{}_nP_p \cdot {}_{n-p}P_p \cdots {}_{n-p_1p}P_p}{p^{n_1}(p-1)^{n_1}}$$

ways. But these cyclic groups may be interchanged in $n_1!$ ways with altering the conjugate of I under consideration. Hence, observing that, w the notation of § 1, $n - n_1 p = a_0$, it follows that *the number of conjug of I under G is*

$$\frac{n!}{n_1!\; a_0!\; p^{n_1}(p-1)^{n_1}} .$$

4. *Number of Conjugates of H.* The structure of the normaliser of I in G, the order of which is $n_1!\,a_0!\,p^{n_1}(p-1)^{n_1}$, is now readily de mined. The cyclic group generated by s_i is invariant under a metacyclic gr M_i, of order $p(p-1)$, which involves only the symbols that s_i affects. H K contains the direct product $M = M_1 M_2, \cdots, M_{n_1}$, of order $p^{n_1}(p-1$ Since s_i and s_j are similar substitutions involving different symbols, we find a substitution u_{ij} of order 2, which transforms s_i into s_j, and w involves only the symbols affected by s_i and s_j. The substitutions u_{ij} cle generate a group G_1 which is contained in K and is simply isomorphic the symmetric group of degree n_1. Since G_1 and M have no substitutio common besides identity, and G_1 transforms M into itself, the orde (M, G_1) is $n_1!\,p^{n_1}(p-1)^{n_1}$. Now K evidently contains the symmetric gr N of order $a_0!$ on the symbols $n_1+1, n_1+2, \cdots, n$, which I does

ach term of which will now be discussed separately.

In order to treat the first term of (14), we break it up into four parts:

$$15)\quad 1/pq\pi^2 \int_0^{\epsilon_1}\int_0^{\epsilon_2} = 1/pq\pi^2 \Big[\int_0^{\epsilon_1}\int_0^{\epsilon_2} + \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} + \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2} + \int_{\epsilon_1}^{\pi/2}\int_{\epsilon_2}^{\pi/2}\Big],$$

here each integral has the same integrand.

For the first term on the right-hand side of (15), we have

$$1/\pi^2 \int_0^{\epsilon_1}\int_0^{\epsilon_2} \phi_1(\alpha\beta)\,\frac{\sin(2m-p)\alpha\sin p\alpha}{p\sin^2\alpha}\,\frac{\sin(2n-q)\beta\sin q\beta}{q\sin^2\beta}\,d\alpha d\beta\,|$$

$$\leqq 1/\pi^2 \int_0^{\epsilon_1}\int_0^{\epsilon_2} |\phi_1(\alpha\beta)|\,\frac{d\alpha}{\sin\alpha}\,\frac{d\beta}{\sin\beta} < \tfrac{1}{4}\int_0^{\epsilon_1}\int_0^{\epsilon_2}\frac{|\phi_1(\alpha\beta)|}{\alpha\beta}\,d\alpha d\beta$$

nce

$$\left|\frac{\sin mx}{\sin x}\right| \leqq m \quad\text{and}\quad \frac{x}{\sin x} \leqq \pi/2 \quad \begin{pmatrix} 0 < x \leqq \pi/2 \\ m = 1, 2, \cdots \end{pmatrix}.$$

t follows from the existence of the first integral given in (11) that, given positive δ, we can choose ϵ_1 and ϵ_2 sufficiently small to make this first rm less in absolute value than $\delta/12$.

For the second integral in (15) we have

$$1/pq\pi^2 \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2}\phi_1(\alpha\beta)\,\frac{\sin(2m-p)\alpha\sin p\alpha}{\sin^2\alpha}\,\frac{\sin(2n-q)\beta\sin q\beta}{\sin^2\beta}\,d\alpha d\beta\,|$$

$$< \frac{1}{4q\sin^2\epsilon_2}\int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2}\frac{|\phi_1(\alpha\beta)|}{\alpha}\,d\alpha d\beta.$$

rom the existence of the second integral in (11), it follows that, since ϵ_1 and are now fixed, we can choose n sufficiently large to make this term less in solute value than $\delta/12$. A similar treatment of the third term in (15), d the existence of the third integral in (11), show that it also can, by proper oice of m, be made less in absolute value than $\delta/12$. The last term in (15) n obviously be made less in absolute value than $\delta/12$ for m and n sufficiently rge. Therefore, by proper choice of m and n, the first term of (14) can be ade less in absolute value than $\delta/3$, contingent on the existence of the ingrals in (11).

For the second term in (14), we have *

---

* We use the identity

$$\frac{1}{\pi}\int_0^{\pi/2}\frac{\sin^2\kappa x}{\sin^2 x} = \kappa/2.$$

$$1/pq\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} \psi_1(\alpha)\,\frac{\sin\,(2m-p)\alpha}{\sin^2\alpha}$$

$$= 1/2p\pi \int_0^{\pi/2}\psi_1(\alpha)$$

We know from Pollard's work * that the
assures us that we can make this part o

That the third term of (14) can b
follows from a similar reduction and t
(12).

By an easy reduction, the fourth t

Hence we have

$$\left|\,\frac{\sum\limits_{m-p+1\ \ n-q+1}^{m\ \ n} s_{ij}^{(1)}}{pq} - s^{(1)}\right| < \delta$$

Similar reductions, as pointed out befo
cases of the other $s^{(i)}$, and the lemma is

LEMMA 3: *If the conditions of L*
of the expressions (4) *corresponding to*
s as a limit.

As in Lemma 2, we carry through

For the double Fourier's series, we

$$(16)\qquad 1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2}\{\psi_1(\alpha)+\psi_2(\qquad \underline{\sin(2m}$$

For the first term, since

$$1/\pi\int_0^{\pi/2}\frac{\sin\kappa x}{\sin x}$$

---

* *Loc. cit.*, p. 339.

† Cf. Bierens de Haan, *Nouvelles Tabl*
No. 7.

we have the following reduction:

$$1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} \psi_1(\alpha)\ \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2\alpha}\ \frac{\sin n\beta}{\sin\beta}\ d\alpha d\beta.$$

$$= 1/2p\pi \int_0^{\pi/2} \psi_1(\alpha)\ \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2\alpha}\ d\alpha.$$

It follows from the discussion of an analogous expression in Pollard's pape that the existence of the first integral of (12) is a sufficient condition th this term can be made less in absolute value than $\delta/3$, for m sufficiently larg

For the second term of (16), we have

$$\left| 1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} \psi_2(\beta)\ \frac{\sin^2 m\alpha - \sin^2(m-p)\alpha}{\sin^2\alpha}\ \frac{\sin n\beta}{\sin\beta}\ d\alpha d\beta \right|$$

$$= \left| 1/2\pi \int_0^{\pi/2} \psi_2(\beta)\ \frac{\sin n\beta}{\sin\beta}\ d\beta \right|$$

$$\leqq \tfrac{1}{4} \int_0^{\pi/2} \left| \frac{\psi_2(\beta)}{\beta} \right| \sin n\beta d\beta.$$

To this we apply the Riemann-Lebesgue Theorem,[*] which, together with t existence of the second integral in (12), enables us to infer that we can ma the expression less in absolute value than $\delta/3$.

The third term of (16) we divide as we did the first term of (14):

$$1/p\pi^2 \int_0^{\pi/2}\int_0^{\pi/2} = 1/p\pi^2 \left\{ \int_0^{\epsilon_1}\int_0^{\epsilon_2} + \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2} + \int_{\epsilon_1}^{\pi/2}\int_{\epsilon_2}^{\pi/2} \right.$$

$$\left. + \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \right\},$$

each integral having the integrand of (16). The discussions of the fi three terms of the right-hand side of the above equation are entirely analogo to those used in the cases of similar terms in Lemma 2. For the last ter

$$\left| 1/p\pi^2 \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \phi_1(\alpha\beta)\ \frac{\sin(2m-p)\alpha \sin p\alpha}{\sin^2\alpha}\ \frac{\sin n\beta}{\sin\beta}\ d\alpha d\right.$$

$$\leqq \frac{1}{4\sin\epsilon_2} \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2} \left| \frac{\phi_1(\alpha\beta)}{\alpha} \right| \sin(2m-p)\alpha \sin n\beta d\alpha d$$

That this tends to zero, we infer from the generalized Riemann-Lebesgue Th orem and the existence of the second integral in (11). Hence the third ter of (16) can be made less in absolute value than

---

[*] Lebesgue, "Sur les Séries Trigonometriques," *Annales Scientifiques de l'Ec normale supérieure,* Ser. 3, Vol. 20 (1903), pp. 471-473.

The fourth term reduces immediately into

Summarizing, we have

$$\lim_{m,n\to\infty} \frac{\sum_{i=m-p+1}^{m} s_{in}{}^{(1)}}{p} =$$

Since the discussions for the other $s^{(i)}$ are simi

As we noted before, the other expressions double Fourier's series can be treated by the n Lemma 2. Hence, applications of these lemma II, if we remember that the other conditions of out as fulfilled by the double Fourier's series.

§ 3. *Generalization to Three Variables of The*

Consider now any infinite triple series $\Sigma\mu$ terms of either sign, and in which

$$(17)\quad \begin{cases} \lim\limits_{lmn\to\infty} \mu_{lmn} = 0 & \\ \lim\limits_{l\to\infty} \mu_{lmn} = 0 \text{ (for every } m, n), & \underset{l,m}{\mathrm{l}} \\ \lim\limits_{m\to\infty} \mu_{lmn} = 0 \text{ (for every } l, n), & \underset{l,n}{\mathrm{l}} \\ \lim\limits_{n\to\infty} \mu_{lmn} = 0 \text{ (for every } l, m), & \underset{m,n}{\mathrm{l}} \end{cases}$$

As in § 1, we construct an auxiliary series

$$(18)\qquad \mu_{lmn} \geqq -a_{lmn}$$

The analogue of Lemma 1 for three varia

LEMMA 4: *If sequences of positive integ be found such that $p_l\to\infty$ steadily with l, while conditions on the $\{q_m\}$ and $\{r_n\}$, and if*

$$\lim_{l,m,n\to\infty} p_l q_m r_n\, a_{lmn} = 0,$$

then

$$\lim_{l,m,n\to\infty} p_l q_m r_n\, a_{l-p_l,\, m-q_m,\, n-r_n} =$$

The proof of this lemma follows at once a Lemma 1.

We are now in a position to deduce the an

THEOREM III: *If the series $\sum \mu_{lmn}$, satisfying conditions (17), is also summable (C1) to s, and if sequences $\{p_l\}$, $\{q_m\}$, and $\{r_n\}$ can be found satisfying the conditions of Lemma 4; and if also each of the expressions of the types**

$$(19)\qquad \frac{\sum\limits_{l-p_l+1,\ m-q_m+1,\ n-r_n+1}^{l,m,n} s_{ijk}}{p_l q_m r_n}\ ;\quad \frac{\sum\limits_{i=l-p_l+1,\ j=m-q_m+1,}^{l,m} s_{ijn}}{p_l q_m}\ ;\quad \frac{\sum\limits_{i=l-p_l+1,}^{l} s_{imn}}{p_l}$$

approaches s as l, m, and n become infinite; then the series converges to s.

By a reduction entirely analogous to that used in § 1 to arrive at equation (5), we have

$$(20)\qquad \begin{aligned} s_{lmn} \geqq\ & \frac{\sum\limits_{i=l-p_l+1}^{l} s_{imn}}{p_l} + \frac{\sum\limits_{j=m-q_m+1}^{m} s_{ljn}}{q_m} + \frac{\sum\limits_{k=n-r_n+1}^{n} s_{lmk}}{r_n} \\ & - \frac{\sum\limits_{i=l-p_l+1,\ j=m-q_m+1}^{l,m} s_{jn}}{p_l q_m} - \frac{\sum\limits_{i=l-p_l+1,\ k=n-r_n+1}^{l,n} s_{imk}}{p_l r_n} - \frac{\sum\limits_{j=m-q_m+1,\ k=n-r_n+1}^{m,n} s_{ljk}}{q_m r_n} \\ & + \frac{\sum\limits_{l-p_l+1,\ m-q_m+1,\ n-r_n+1}^{l,m,n} s_{ijk}}{p_l q_m r_n} - p_l q_m r_n\, a_{l-p_l,\ m-q_m,\ n-r_n}\,. \end{aligned}$$

Similarly,

$$(21)\qquad \begin{aligned} \frac{\sum\limits_{l+1,\ m+1,\ n+1}^{l+p_l,\ m+q_m,\ n+r_n} s_{ijk}}{p_l q_m r_n} \geqq\ & \frac{\sum\limits_{i=l+1,\ j=m+1}^{l+p_l,\ m+q_m,} s_{ijm}}{p_l q_m} + \frac{\sum\limits_{i=l+1,\ k=n+1}^{l+p_l,\ n+r_n} s_{imk}}{p_l r_n} + \frac{\sum\limits_{j=m+1,\ k=n+1}^{m+q_m,\ n+r_n} s_{ljk}}{q_m r_n} \\ & - \frac{\sum\limits_{i=l+1}^{l+p_l} s_{imn}}{p_l} - \frac{\sum\limits_{j=m+1}^{m+q_m} s_{ljn}}{q_m} - \frac{\sum\limits_{k=n+1}^{n+r_n} s_{lmk}}{r_n} \\ & + s_{lmn} - p_l q_m r_n\, a_{lmn}\,. \end{aligned}$$

As *l, m,* and *n* become infinite, Lemma 4 and the hypothesis enable us to deduce from (20) that

$$(22)\qquad \underline{\lim}\ s_{lmn} \geqq s;$$

and from (21), that

$$(23)\qquad \overline{\lim}\ s_{lmn} \leqq s.$$

A combination of (22) and (23) gives

---

* The other eleven expressions of these types appear in the body of the proof of the theorem.

$$(24) \qquad \lim_{l,m,n\to\infty} s_{lmn} = s,$$

and the theorem is proved.

§ 4. *Application to the Triple Fourier's Series.*

Theorem III will now be applied to the triple Fourier's series, to a condition for convergence analogous to the Dini conditions mentioned and to the generalizations of those conditions obtained in § 2.

We first write for convenience:

$$\begin{aligned}
\phi(a,\beta,\gamma) \equiv{}& f(x+2a, y+2\beta, z+2\gamma) - f(x+2a, y+2\beta, z+0) \\
&\qquad\qquad - f(x+2a, y+0, z+ \\
&- f(x+0, y+2\beta, z+2\gamma) + f(x+2a, y+0, z+0) \\
&\qquad\qquad + f(x+0, y+2\beta, z \\
&+ f(x+0, y+0, z+2\gamma) - f(x+0, y+0, z+0) \\
\chi_1(a,\beta) \equiv{}& f(x+2a, y+2\beta, z+0) - f(x+0, y+0, z+0) \\
\chi_2(a,\gamma) \equiv{}& f(x+2a, y+0, z+2\gamma) - f(x+0, y+0, z+0) \\
\chi_3(\beta,\gamma) \equiv{}& f(x+0, y+2\beta, z+2\gamma) - f(x+0, y+0, z+0) \\
\chi_4(a) \equiv{}& f(x+2a, y+0, z+0) - f(x+0, y+0, z+0) \\
\chi_5(\beta) \equiv{}& f(x+0, y+2\beta, z+0) - f(x+0, y+0, z+0) \\
\chi_6(\gamma) \equiv{}& f(x+0, y+0, z+2\gamma) - f(x+0, y+0, z+0) \\
\Phi(a,\beta,\gamma) \equiv{}& \phi(a,\beta,\gamma) + \chi_1(a,\beta) + \chi_2(a,\gamma) + \chi_3(\beta,\gamma) - \chi_4(a) \\
&- \chi_5(\beta) - \chi_6(\gamma) + f(x+0, y+0, z+0) \\
={}& f(x+2a, y+2\beta, z+2\gamma).
\end{aligned}$$

We are then ready to prove the following theorem:

THEOREM IV: *If $f(x,y,z)$ has a Lebesgue integral in the $(-\pi \leqq x \leqq \pi;\ -\pi \leqq y \leqq \pi;\ -\pi \leqq z \leqq \pi)$, and if its developmen triple Fourier's series is summable $(C1)$ to the value*

$$(25) \qquad s = \tfrac{1}{8}\left\{\begin{array}{l} f(x+0, y+0, z+0) + f(x+0, y+0, z-0) + f(x+0, y- \\ + f(x-0, y+0, z+0) + f(x+0, y-0, z-0) + f(x-0 \\ + f(x-0, y-0, z+0) + f(x-0, y-0, z-0) \end{array}\right.$$

at the point (x,y,z); then the development also converges to s at that *provided the integrals of the following types exist:*

$$(26)\quad \left\{\begin{array}{l} \displaystyle\int_0^{\epsilon_1}\int_0^{\epsilon_2}\int_0^{\epsilon_3}\frac{|\phi(\alpha,\beta,\gamma)|}{\alpha\beta\gamma}\,d\alpha d\beta d\gamma \quad \int_{\epsilon_1}^{\pi/2}\int_{\epsilon_2}^{\pi/2}\int_0^{\epsilon_3}\frac{|\phi(\alpha\beta\gamma)|}{\gamma}\,d\alpha \\ \displaystyle\qquad\qquad\int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2}\int_0^{\epsilon_3}\frac{|\phi(\alpha\beta\gamma)|}{\beta\gamma}\,d\alpha d\beta d\gamma \end{array}\right.$$

$$(27)\quad \int_{\epsilon_1}^{\pi/2}\int_0^{\epsilon_2}\frac{|\chi_1(\alpha\beta)|}{\beta}\,d\alpha d\beta \qquad \int_0^{\epsilon_1}\int_{\epsilon_2}^{\pi/2}\frac{|\chi_1(\alpha\beta)|}{\alpha}\,d\alpha d\beta$$

$$(28)\quad \int_0^{\epsilon_1}\frac{|\chi_4(\alpha)|}{\alpha}\,d\alpha$$

for arbitrary, small, positive ϵ's.

As in § 2, sequences $\{p_l\}$, $\{q_m\}$, and $\{r_n\}$, with the desired properties, can be found, since the general term of the triple Fourier's series approaches zero as l, m, and n become infinite.* Hence, to prove our theorem, it remains only to be shown that the expressions of the form (19) for the triple Fourier's series approach s as a limit as l, m and n become infinite. We shall indicate the discussion for the expressions given explicitly in (19), with the remark that the reductions for the others are similar.

We set

$$(29)\qquad \sigma_{lmn} = \frac{1}{lmn\pi^3}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(x+2\alpha, y+2\beta, z+2\gamma) \frac{\sin^2 l\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma}{\sin^2\gamma}\,d\alpha d\beta d\gamma;$$

and introduce a notation analogous to that used in § 2: $\sigma^{(i)}_{lmn}$; $s^{(i)}_{lmn}$; $s^{(i)}$; $(i=1, 2, \cdots, 8)$, where $\sigma^{(i)}_{lmn}$ represents the expressions analogous to (29) in which the integration is extended only over that portion of the original region included in a single octant; and where each $s^{(i)}$ is the limit of a corresponding $\sigma^{(i)}_{lmn}$.

We are then ready to prove three lemmas concerning the behavior of the expressions (19) corresponding to the triple Fourier's series.

LEMMA 5: *If $f(x, y, z)$ satisfies the conditions of the above theorem, the first expression in* (19), *corresponding to its triple Fourier's series, will approach s as a limit.*

Proceeding in exactly the same manner as in Lemma 2, we shall establish the result for $s^{(1)}$, and since the proof for the corresponding expressions with the other upper indices is similar, our lemma follows immediately.

---

* This follows from a further generalization of the Riemann-Lebesgue Theorem.

For the triple Fourier's series, we can write this expression as

$$\frac{1}{pqr\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} f(x+2\alpha, y+2\beta, z+2\gamma)$$

$$\frac{\sin^2 l\alpha-\sin^2(l-p)\alpha}{\sin^2\alpha}\,\frac{\sin^2 m\beta-\sin^2(m-q)\beta}{\sin^2\beta}\,\frac{\sin^2 n\gamma-\sin^2(n-r)\gamma}{\sin^2\gamma}\,d\alpha d\beta d\gamma,$$

which reduces to

$$(30)\qquad \frac{1}{pqr\pi^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\Phi(\alpha\beta\gamma)$$

$$\frac{\sin(2l-p)\alpha\sin p\alpha}{\sin^2\alpha}\,\frac{\sin(2m-q)\beta\sin q\beta}{\sin^2\beta}\,\frac{\sin(2n-r)\gamma\sin r\gamma}{\sin^2\gamma}\,d\alpha d\beta d\gamma;$$

and for the purposes of the lemma, we break it up into eight terms corresponding to the eight different terms of $\Phi(\alpha, \beta, \gamma)$.

We note first that the last of these terms reduces immediately to $s^{(1)}$. The treatment of the first term is an obvious extension of the discussion of the first term in (14); and the others can be reduced immediately to expressions whose treatments are covered by the various discussions in Lemma 2 or in Pollard's paper. Hence it follows from the existence of (26), (27), and (28) that, by proper choice of l, m and n, each of the terms we are considering can be made less in absolute value than $\delta/7$; therefore

$$\lim_{l,m,n\to\infty}\frac{\sum\limits_{l-p+1,\ m-q+1,\ n-r+1}^{l,\ m,\ n} s^{(1)}{}_{ijk}}{pqr} = s^{(1)}.$$

Since similar reductions hold for the other $s^{(i)}$, the lemma follows readily.

LEMMA 6: *If the conditions of Lemma 5 are satisfied, then the second of the expressions* (19) *corresponding to the triple Fourier's series will approach s as a limit.*

LEMMA 7: *If the conditions of Lemma 5 are satisfied, then the third of the expressions* (19) *corresponding to the triple Fourier's series approaches s as a limit.*

The proofs of these lemmas are entirely analogous to those outlined in the other lemmas of the paper; these lemmas, then, follow readily, contingent on the existence of the integrals (26), (27), and (28).

The theorem of this section is an immediate consequence of the three lemmas just stated, since it follows from these lemmas that the expressions (19) have the desired properties. Hence, a sufficient condition for the convergence of a summable triple Fourier's series is the existence of the integrals given in (26), (27), and (28), which are the generalizations to three variables of the Dini condition for the convergence of the simple Fourier's series.

UNIVERSITY OF CINCINNATI,
JUNE, 1924.

On the Power Characters of Units in a Cyclotomic Field.

By H. S. Vandiver.

If we consider the relative cyclotomic field $\Omega(K)$, that is, an algebraic field which contains the cyclotomic sub-field $\Omega(\alpha)$, $\alpha = e^{2i\pi/p}$, p an odd prime, then if ω is an integer in $\Omega(K)$ and $\mathfrak{q}$ is an ideal prime, we have

$$\omega^{N(\mathfrak{q})-1} \equiv 1 \pmod{\mathfrak{q}}$$

where $N(\mathfrak{q})$ is the norm of $\mathfrak{q}$ in $\Omega(K)$, and (ω) is prime to $\mathfrak{q}$. Also there is one and but one integer a in the set $0, 1, 2, \cdots, p-1$, such that

$$\omega^{\frac{N(\mathfrak{q})-1}{p}} \equiv \alpha^a \pmod{\mathfrak{q}}$$

provided $\mathfrak{q}$ is prime to p. The expression α^a is known as the power character of ω with respect to $\mathfrak{q}$ in the field $\Omega(K)$. We also write

$$\left\{ \frac{\omega}{\mathfrak{q}} \right\} = \alpha^a.$$

These power characters have been treated by Furtwängler* who gave two laws of reciprocity involving them. Consider the problem of determining $(\omega/\mathfrak{q})$, where ω is a *unit* in $\Omega(K)$. For the special case where $\Omega(K)$ is the field $\Omega(\alpha)$ itself, the power character was determined by Kummer† in terms of known functions of the ideal $\mathfrak{q}$, provided $\Omega(\alpha)$ is a regular field, and ω is any unit of the type

$$e(\alpha) = \sqrt{\frac{(1-\alpha^{\gamma})(1-\alpha^{-\gamma})}{(1-\alpha)(1-\alpha^{-1})}}.$$

The units $e(\alpha)$, $e(\alpha^{\gamma}) \cdots$, $e(\alpha^{\gamma^{\mu-2}})$ (where $\mu = (p-1)/2$, and γ is a primitive root of p) form an independent system, so that any unit in $\Omega(\alpha)$ can be expressed in the form

$$e(\alpha)^{n_0},\ e(\alpha^{\gamma})^{n_1},\ \cdots,\ e(\alpha^{\gamma^{\mu-2}})^{n_{\mu-2}}$$

the n's being rational fractions.

---

† *Crelle*, 44, 1852, 93-146.

* Cf. references to Furtwängler in "The Report on the Theory of Algebraic Numbers," *Bull. Natl. Research Council*, Feb. 1923, p. 20.

In particular if

$$E_n(\alpha) = e(\alpha) \ e(\alpha\gamma)^{\gamma^{-2n}} \ e(\alpha\gamma^2)^{\gamma^{-4n}} \cdots e(\alpha\gamma^{\mu-1})^{\gamma^{-2(\mu-1)n}}$$

then Kummer * proved the relation, where λ is written in place of p, and λ is any odd prime,

$$(1) \quad a \equiv \frac{\gamma^{2n} - 1}{2(1 + r^{\lambda-2n} - (r+1)^{\lambda-2n})} \cdot \frac{d_0{}^{\lambda-2n} \log \Psi_r(e^v)}{dv^{\lambda-2n}} \pmod{\lambda}$$

where

$$\left\{ \frac{E_n(\alpha)}{\mathfrak{P}} \right\} = \alpha^a,$$

r is any integer $\not\equiv 0$ or -1, modulo λ,

$$\Psi_r(x) = \sum_h x^{-(r+1)h + \operatorname{ind}(g^h+1)}$$

$\mathfrak{P}$ is an ideal prime in $\Omega(\alpha)$ whose norm is q^t, q odd, g is a primitive root of $\mathfrak{P}$, h ranges over the integers $0, 1, 2, \cdots, q^t-2$, excepting $(q^t-1)/2$, and g is selected so that $g^{(q^t-1)/\lambda} \equiv \alpha \pmod{\mathfrak{P}}$. The symbol

$$\frac{d_0{}^{\lambda-2n} \log \Psi_r(e^v)}{dv^{\lambda-2n}}$$

means that the $(\lambda - 2n)$th derivative of $\log \Psi_r(e^v)$ is taken with respect to v, and $v = 0$ substituted in the result. Here e is the Napierian base. Owing to certain applications I expect to make in another paper, it will be necessary to use an extension of Kummer's results to the fields defined by primitive pnth roots of unity, n prime to p, and the present paper will be devoted to the proof of the extended theorem.

I was enabled to carry out this investigation through a grant, relieving me of the duties of teaching from the Heckscher foundation for the Advancement of Research, established by August Heckscher at Cornell University.

1. *Transformations depending on the Jacobi-Kummer cyclotomic function.*

This function was treated by H. H. Mitchell † and is defined as

$$\Psi_{a,b}(\theta) = \sum_h \theta^{-bh + (a+b)\operatorname{ind}(g^h+1)}$$

where θ is now any primitive mth root of unity ($m = pn$) ind ω is determined by $g^{\operatorname{ind}\omega} \equiv \omega \pmod{\mathfrak{q}}$, $\mathfrak{q}$ being an ideal prime in $\Omega(\theta)$ whose norm is q^t, g is a primitive root of $\mathfrak{q}$, and h ranges over $0, 1, \cdots, q^t - 2$, with the exception of $(q^t - 1)/2$ if q is odd and zero if $q = 2$. Also g is selected so that

---

* *l. c.*, 106-125.

† *Transactions Amer. Math. Soc.*, 17, 1916, 165-177

$$g^{\frac{q^t-1}{m}} \equiv \theta \text{ (mod. q)}.$$

As defined here this function is a special case of that treated by Mitchell. It has the properties (Mitchell, *l. c.*, p. 168)

$$\Psi(\theta)\,\Psi(\theta^{-1}) = q^t \tag{2}$$

if $a \not\equiv 0$, $b \not\equiv 0$, $a + b \not\equiv 0$ (mod. m), and

$$\Psi_{a,\,0}(\theta) = \Psi_{0,\,b}(\theta) = -1. \tag{3}$$

We now set, if $\beta = e^{2i\pi/n}$

$$D_r = \frac{d^r \log \Psi\,(e^v\beta)}{dv^r}$$

and

$$D_r(\beta) = \left[\frac{d^r \log \Psi\,(e^v\beta)}{dv^r}\right]_{v=0}$$

this symbol indicating that $v = 0$ is substituted in the function of e^v after differentiation. Put $U = 1/\Psi(e^v\beta)$, we have

$$\frac{d \log \Psi(e^v\beta)}{dv} = U\,\frac{d\,\Psi(e^v\beta)}{dv}$$

and

$$D_r = \frac{d^r\,\Psi(e^v\beta)}{dv^j}\,U + (r-1)\,\frac{d^{r-1}\,\Psi(e^v\beta)}{dv^{j-1}} \cdot \frac{dU}{dv} + \cdots + \frac{d\,\Psi(e^v\beta)}{dv} \cdot \frac{d^{r-1}U}{dv}. \tag{4}$$

Now by (2) the function

$$A = \Psi(x\beta)\,\Psi(x^{-1}\beta^{-1}) - q^t$$

vanishes for $x = \alpha$. Consequently since $(x^p - 1)/(x - 1)$ is irreducible in the field $\Omega(\beta)$,* we have

$$A = W(x)\,\frac{x^p - 1}{x - 1}$$

where $W(x)$ is an integral function of x, with coefficients in the field $\Omega(\beta)$. If $V = (e^{vp} - 1)/(e^v - 1)$ we have

$$\Psi(e^v\beta)\,\Psi(e^{-v}\beta^{-1}) = q^t + VW(e^v) \tag{5}$$

---

* Weber, *Algebra*, Vol. 1, 2nd ed., 600.

and

$$\Psi(e^{-v}\beta^{-1}) = \frac{q^t + VW}{\Psi(e^v)} = (q^t + VW)U,$$

where $W = W(e^v)$.

We shall now transform this expression by means of variations and extensions of the methods employed by Kummer[†] for the special case in which θ is a primitive pth root of unity.

We shall use the notation

$$F(e^v\beta)^{(i)} = \frac{d^i F(e^v\beta)}{dv^i}$$

and

$$F(e^v\beta)_0^{(i)} = \left[\frac{d^i F(e^v\beta)}{dv^i}\right]_{v=0},$$

$$(6) \qquad \Psi(e^{-v}\beta^{-1})^{(i)} = U^{(i)}(q^t + VW) + iU^{(i-1)}(VW)^1 + \cdots$$

For $v = 0$ this reduces, since $q^t \equiv 1 \pmod{m}$ to

$$\Psi(e^{-v}\beta^{-1})_0^{(i)} \equiv U_0^{(i)} \pmod{p}$$

$i = 0, 1, 2, \cdots, p-2$.

(4) then gives

$$D_r(\beta) \equiv \Psi(e^v\beta)^r\,\Psi(e^{-v}\beta^{-1}) + (r-1)\,\Psi(e^v\beta)^{(r-1)}\,\Psi(e^{-v}\beta^{-1})^{(1)} \\ + \cdots \pmod{p}.$$

If

$$\Psi(e^v\beta)_0^{(i)} = f_i(\beta)$$

then

$$\Psi(e^{-v}\beta^{-1})_0^{(i)} \equiv (-1)^i f_i(\beta^{-1})$$

and therefore

$$D_r(\beta) \equiv f_r(\beta)f_0(\beta^{-1}) - (r-1)f_{r-1}(\beta)f_1(\beta^{-1}) \\ + \cdots - f_1(\beta)f_{r-1}(\beta^{-1}) \pmod{p}.$$

Now

$$f_i(\beta) = \sum_h (-bh + (a+b)\text{ ind }(g^h+1))^i\beta^{-bh+\text{ind}(g^h+1)}$$

h ranging over the values stated in definition.

Set

$$-bh + (a+b)\text{ ind }(g^h+1) = C_h,$$

$$\text{and } -bk + (a+b)\text{ ind }(g^k+1) = C_k.$$

Then

$$D_r(\beta) \equiv \sum_h \sum_k (C_h{}^r C_k{}^0 - (r-1)C_h{}^{r-1}C_k + \cdots)\beta^{C_h - C_k} \pmod{p},$$

[†] *Journal für Mathematik*, Vol. 44, p. 122.

or

$$(7)\qquad D_r(\beta) \equiv \sum_h \sum_k C_h(C_h - C_k)^{r-1}\beta^{C_h - C_k} \pmod{p}$$

where h and k each range over the integers $0, 1, \cdots, q^t - 2$, excepting $(q^t - 1)/2$ if q is odd and zero if $q = 2$. Put

$$g^{k-h} \equiv g^{k'} \frac{g^k + 1}{g^h + 1} \equiv g^{k'} \pmod{q}.$$

Then the right hand member of (7) may be written

$$\sum_{k^1} \sum_{h^1} (-b \text{ in } (g^{h'} - 1) + (a + b) \text{ ind } (g^{k'} - 1) - a \text{ ind } (g^{k'} - g^{h'}))$$
$$\times (bk' - (a + b)h')^{r-1} \beta^{bk' - (a+b)h'}$$

where h' and k' each range over $1, 2, \cdots, q^t - 2$, but $h' \neq k'$, since in that case ind $(g^{k'} - g^{h'})$ has no meaning. Set, if $pl \equiv -1 \pmod{n}$,

$$(8)\qquad \begin{cases} a = pl \\ b = n - pl. \end{cases}$$

Evidently these values satisfy the restrictions on relation (2). Applying them to the last expression, it reduces modulo p, to

$$n^r \sum_k \sum_h (-\text{ind } (g^{h'} - 1) + \text{ind } (g^{k'} - 1)(k' - h')^{r-1} \beta^{k'}$$

We re-introduce the values $h' = k'$ since for such values the expression vanishes. We again use the symbols h for h' and k for k'. We shall now employ the generalized function treated by Mitchell.

Let $\theta_1 = e^{2i\pi/n_1 p}$ where n_1 is any divisor of n other than unity. Then using the same Ψ function as before, we have

$$\Psi(\theta_1)\Psi(\theta_1^{-1}) = q^t$$

since (8) gives

$$a \not\equiv 0, \quad b \not\equiv 0, \quad a + b \not\equiv 0 \pmod{n_1 p}.$$

We now go thru exactly the same argument with this relation as with (5). There is obtained

$$D_r(\beta_1) \equiv n^r \sum_k \sum_h (-\text{ind } (g^h - 1) + \text{ind } (g^k - 1))(k - h)^{r-1} \beta_1^k$$

modulo p, if $\beta_1 = e^{2i\pi/n_1}$. Since n_1 is any divisor of n we can then write

$$D_r(\beta^s) \equiv n^r \sum_k \sum_h (-\text{ind } (g^h - 1) + \text{ind } (g^k - 1))(k - h)^{r-1} \beta^{ks}$$

$$s = 1, 2, \cdots, n - 1.$$

Since $\sum_{i=1}^{n-1} \beta^{ji} = (\beta^{nj} - 1)/(\beta^{j} - 1) - 1 = -1$, $j \not\equiv 0 \pmod{n}$, then

$$(9) \qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s)$$

$$\equiv -n^r \sum_k \sum_h (-\operatorname{ind}(g^h - 1) + \operatorname{ind}(g^k - 1)(k - h)^{r-1}$$

$$+ n^{r+1} \sum_K \sum_h (-\operatorname{ind}(g^h - 1) + \operatorname{ind}(g^K - 1))(K - h)^{r-1}$$

c being a particular integer in the set $1, 2, \cdots, n-1$, and K ranges over the integers $< q^t - 1$ which satisfy $K \equiv c \pmod{n}$.

We have

$$\sum_k \sum_h (-\operatorname{ind}(g^h - 1) + \operatorname{ind}(g^k - 1))(k - h)^{r-1}$$

$$= -\sum_k \sum_h \operatorname{ind}(g^h - 1)(k - h)^{r-1} + \sum_k \sum_h \operatorname{ind}(g^k - 1)(k - h)^{r-1}$$

$$\equiv \sum_k - (q^t - 2)(-h)^{r-1} \operatorname{ind}(g^h - 1)$$

$$+ \sum_k (q^t - 2) k^{r-1} \operatorname{ind}(g^k - 1) \pmod{p},$$

since $\sum_k h^s \equiv 0 \pmod{p}$, $s = 1, 2, \cdots, p-2$.

Noting that $q^t \equiv 1 \pmod{p}$, the above reduces to

$$((-1)^{r-1} - 1) \sum_h h^{r-1} \operatorname{ind}(g^h - 1) \pmod{p}.$$

Similarly

$$-\sum_K \sum_h \operatorname{ind}(g^h - 1)(K - h)^{r-1}$$

$$+ \sum_K \sum_h \operatorname{ind}(g^K - 1)(K - h)^{r-1}$$

reduces to

$$-\sum_K K^{r-1} \operatorname{ind}(g^K - 1) \pmod{p},$$

if we observe that $\sum_K K^s \equiv 0 \pmod{p}$, $s = 1, 2, \cdots, p-2$, and that there are $(q^t - 1)/n$ values of K. Hence (9) becomes, if $r < p$,

$$(10) \qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s)$$

$$\equiv n^r((-1)^r + 1) \sum_h h^{r-1} \operatorname{ind}(g^h - 1)$$

$$- n^{r+1} \sum_K K^{r-1} \operatorname{ind}(g^K - 1) \pmod{p}.$$

Now if

$$\nu = (q^t - 1)$$

$$\sum_h h^{r-1} \operatorname{ind} (g^h - 1) \equiv \sum_j \sum_i$$

where $i = 1, 2, \cdots, m - 1$, and $j =$
But

$$(g^i - 1)(g^{i+m} - 1) \cdots (g$$

whence, noting that ind $(-1) \equiv 0$ (

$$\sum_j \operatorname{ind} (g^{i+jm} - 1) \equiv \text{in}$$

and since $g^\nu \equiv \theta$ (mod. q), then

$$\sum_j \operatorname{ind} (g^{i+jm} - 1) \equiv \text{in}$$

and (10) then becomes, for $r < p$,

$$\text{(11)} \qquad \sum_{s=1}^{n-1} \beta^{-cs} D_r(\beta^s) \equiv$$
$$((-1)^r + 1) \sum_i i^{r-1} \operatorname{ind}$$

where $i = 1, 2, \cdots, m - 1$, and l
$l \equiv c$ (mod. n).

2. If ω is any integer in $\Omega(\theta)$,

$$\omega^n \equiv g^n$$

$$\omega^{\frac{q^t-1}{m} \cdot n} \equiv g^{\frac{q^t-1}{m}}$$

or, since $g^{(q^t-1)/m} \equiv \theta$ (mod. q)

$$\omega^{\frac{q^t-1}{p}} \equiv \theta^{n \operatorname{ind} \omega} \equiv a^n$$

and

$$\Big\{ \frac{\omega}{\mathfrak{q}}$$

Writing $I(\omega)$ for n ind ω we have f

$$\text{(11a)} \qquad \sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s) \equiv n^{r-1}((-$$

Let $i = ep + d$. If d is fixed, then e ranges over the integers $0, 1, 2, \cdots, n-1$ and then

$$\prod_e (\theta^{ep+d} - 1) = \theta^{nd} \prod_e (\theta^{ep} - \theta^{-d})$$
$$= \theta^{nd} - 1$$

and therefore

$$\sum_i i^{r-1} I(\theta^i - 1) \equiv \sum_d d^{r-1} I(\alpha^{nd} - 1) \pmod{p},$$

$d = 1, 2, \cdots, p-1$; and this may be written

$$\sum_{d=1}^{p-1} d^{r-1} I(\alpha^{nd} - 1) \equiv \sum_{d_1=1}^{(p-1)/2} d_1^{r-1} I(\alpha^{nd_1/2})$$
$$+ \sum_{d_1} I(\alpha^{nd_1/2} - \alpha^{nd_1/2})$$
$$+ \sum_{d_1} (p - d_1)^{r-1} ((I(\alpha^{-nd_1/2}) + I(\alpha^{nd_1/2} - \alpha^{-nd_1/2}))$$

and since the i term exists only in (11a) when r is even, we have

$$\sum_{d=1}^{p-1} d^{r-1} I(\alpha^{nd} - 1) \equiv n I(\alpha) \sum_{d_1=1}^{(p-1)/2} d_1^{r} \pmod{p}$$

and since r is even then

$$\sum_{d_1} d_1^{r} \equiv 0 \pmod{p}$$

$r = 2, 4, 6, \cdots, p-3$, and

$$\sum_{d_1} d_1^{p-1} \equiv (p-1)/2 \pmod{p},$$

so that from (11a) we have

$$\sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s) \equiv - n^r \sum_l l^{r-1} I(\theta^l - 1) \pmod{p}$$

$0 < r < p-1$, and

$$\sum_{s=1}^{n-1} \beta^{-sc} D_{p-1}(\beta^s) \equiv - I(\alpha) - \sum_l l^{p-2} I(\theta^l - 1) \pmod{p},$$

or we may write

$$\sum_{s=1}^{n-1} \beta^{-sc} D_r(\beta^s) \equiv - n^r \sum_{l=1}^{p-1} I(\alpha^l \beta^c - 1) \pmod{p} \tag{12}$$

$0 < r < p-1$, and

$$\sum_{s=1}^{n-1} \beta^{-sc} D_{p-1}(\beta^s) \equiv - I(\alpha) - \sum_{l=1}^{p-1} l^{p-2} I(\alpha^l \beta^c - 1) \pmod{p}. \tag{13}$$

These two relations are the extensions of Kummer's relation (1).

Sec. No.........
Book No..........
LIBRARY.

A Configuration of Thirteen Pencils of Cubics and Cubics with Three Real Inflexions.

BY B. M. TURNER.

In an earlier paper (P_1)* the writer discussed the system of plane cubics with a given quadrangle of inflexions. The nine points of inflexion of any one cubic offer a number of ways of choosing four points which form a quadrangle, but among these quadrangles some have vertices in common. It follows that all the cubics with a given quadrangle of inflexions in common with one inflexional set form a configuration with notable geometric properties. The first part of this paper gives the results of a study of the real cubics contained in this configuration. The second part is devoted to a discussion of cubics with three real inflexions given.

I. *Thirteen Pencils of Cubics.*

From among the nine points of inflexion of a real non-singular cubic there are three ways of selecting four points, two real and one pair imaginary, no three collinear; and three ways of selecting two imaginary pairs. In this paper quadrangles with two real vertices are considered first because they lead to a geometrical construction complete in itself, while the corresponding construction from two pairs of imaginary points serves only as a supplement to the former in a more extensive complete figure. For convenience in the investigation, the canonical inflexional set of Hesse is taken as the oirgin.

§ 1. Quadrangles of Inflexions: Two Real and One Pair Imaginary.

As shown in P_1†, every set of four points, two real and one pair imaginary, no three collinear, belong to two, and only two, completely determined inflexional sets for real cubics. Hence, the three such sets of four points selected from

$$
\text{(A)} \quad \begin{array}{lll}
(0,\ 1,\ -1), & (0,\ 1,\ -\omega), & (-,\ 1,\ -\omega^2), \\
(-\omega^2,\ 0,\ 1), & (-1,\ 0,\ 1), & (-\omega,\ 0,\ 1), \\
(1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (1,\ -1,\ 0),
\end{array}
$$

---

* *American Journal of Mathematics,* Vol. XLIV, No. 4, Oct. 1922. Throughout this article the earlier paper is referred to as P_1.

† P_1, pages 266, 275.

determine uniquely the three following inflexional sets:

$$\begin{array}{llll}
 & (0,\ 1,\ -1), & (\omega-1,\ -1,\ 1), & (\omega^2-1,\ -1,\ 1), \\
(1) & (-1,\ \omega-1,\ 1), & (-1,\ 0,\ 1), & (-1,\ \omega^2-1,\ 1), \\
 & (1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (1,\ -1,\ 0); \\
 & (0,\ 1,\ -1), & (\omega^2-1,\ 1,\ -1), & (\omega-1,\ 1,\ -1), \\
(2) & (-\omega^2,\ 0,\ 1), & (-1,\ 0,\ 1), & (-\omega,\ 0,\ 1), \\
 & (-1,\ 1,\ \omega-1), & (-1,\ 1,\ \omega^2-1), & (1,\ -1,\ 0); \\
 & (0,\ 1,\ -1), & (0,\ 1,\ -\omega), & (0,\ 1,\ -\omega^2), \\
(3) & (1,\ \omega^2-1,\ -1), & (-1,\ 0,\ 1), & (1,\ \omega-1,\ -1), \\
 & (1,\ -1,\ \omega^2-1), & (1,\ -1,\ \omega-1), & (1,\ -1,\ 0).
\end{array}$$

Each of the three derived sets has, in addition to the four selected points, the third real point in common with (A). Hence the following theorem:

(1) There are three, and only three real pencils of cubics with three real and a pair of imaginary points of inflexion in common with a given real pencil.

These four inflexional sets contain only twenty-one distinct points:—namely, the nine of (A) and twelve others related to the nine by equianharmonic properties. The four pencils of cubics have only seven real inflexional axes, the four of the pencil corresponding to the set (A),

$$x=0, \quad y=0, \quad z=0, \quad x+y+z=0,$$

forming a quadrilateral and the sides of the accompanying diagonal triangle,

$$y+z=0, \quad z+x=0, \quad x+y=0.$$

The twenty-one points are the intersections of the seven real axes by the imaginary lines

$$\begin{array}{ll}
\omega x+y+z=0, & \omega^2x+y+z=0; \\
x+\omega y+z=0, & x+\omega^2y+z=0; \\
x+y+\omega z=0, & x+y+\omega^2z=0.
\end{array}$$

Thus:

(2) The twenty-one points of inflexion of a given real syzygetic pencil of cubics and the three real pencils having in common with it three real and a pair of imaginary inflexions, all lie on seven real lines which form a quadrilateral and its diagonal triangle, and are determined by the three pairs of imaginary inflexional axes of the given pencil which pass through the three real inflexions.

§ 2. Quadrangle of Inflexions: Two Imaginary Pairs.

Two pairs of imaginary points which form a quadrangle belong to four completely determined inflexional sets for real cubics.[*] When the three such quadrangles are selected from

$$\begin{array}{llll}
 & (0,\ 1,\ -1), & (0,\ 1,\ -\omega), & (0,\ 1,\ -\omega^2), \\
(A) & (-\omega^2,\ 0,\ 1), & (-1,\ 0,\ 1), & (-\omega,\ 0,\ 1), \\
 & (1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (1,\ -1,\ 0),
\end{array}$$

the nine derived sets are the following:

$$\begin{array}{llll}
 & (0,\ 1,\ -1), & (1-\omega^2,\ 1,\ -\omega^2), & (1-\omega,\ 1,\ -\omega), \\
(1) & (-\omega^2,\ 0,\ 1), & (2,\ 0,\ 1), & (-\omega,\ 0,\ 1), \\
 & (1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (2,\ 1,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (1,\ 1,\ 1), & (1,\ 1,\ \omega^2), & (1,\ 1,\ \omega), \\
(2) & (-\omega,\ 0,\ 1), & (-1,\ 0,\ 1), & (-\omega^2,\ 0,\ 1), \\
 & (1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (2,\ 1,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (1,\ 1,\ 1), & (1,\ \omega^2,\ 1), & (1,\ \omega,\ 1), \\
(3) & (-\omega,\ 0,\ 1), & (2,\ 0,\ 1), & (-\omega^2,\ 0,\ 1), \\
 & (1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (1,\ -1,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (0,\ 2,\ 1), & (0,\ 1,\ -\omega), & (0,\ 1,\ -\omega^2), \\
(4) & (-\omega,\ 1-\omega,\ 1), & (-1,\ 0,\ 1), & (-\omega^2,\ 1-\omega^2,\ 1), \\
 & (1,\ -\omega,\ 0), & (1,\ -\omega^2,\ 0), & (1,\ 2,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (0,\ 2,\ 1), & (0,\ 1,\ -\omega), & (0,\ 1,\ -\omega^2), \\
(5) & (\omega,\ 1,\ 1), & (1,\ 1,\ 1), & (\omega^2,\ 1,\ 1), \\
 & (1,\ -\omega^2,\ 0), & (1,\ -\omega,\ 0), & (1,\ -1,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (0,\ 1,\ -1), & (0,\ 1,\ -\omega), & (0,\ 1,\ -\omega^2), \\
(6) & (1,\ 1,\ \omega), & (1,\ 1,\ 1), & (1,\ 1,\ \omega^2), \\
 & (1,\ -\omega^2,\ 0), & (1,\ -\omega,\ 0), & (1,\ 2,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (0,\ 1,\ 2), & (0,\ 1,\ -\omega), & (0,\ 1,\ -\omega^2), \\
(7) & (-\omega^2,\ 0,\ 1), & (1,\ 0,\ 2), & (-\omega,\ 0,\ 1), \\
 & (1,\ -\omega^2,\ 1-\omega^2), & (1,\ -\omega,\ 1-\omega), & (1,\ -1,\ 0);
\end{array}$$

$$\begin{array}{llll}
 & (0,\ 1,\ -1), & (0,\ 1,\ -\omega^2), & (0,\ 1,\ -\omega), \\
(8) & (-\omega^2,\ 0,\ 1), & (1,\ 0,\ 2), & (-\omega,\ 0,\ 1), \\
 & (1,\ \omega^2,\ 1), & (1,\ \omega,\ 1), & (1,\ 1,\ 1);
\end{array}$$

---

[*] P_1, page 264.

$$(9)\qquad \begin{array}{lll} (0,\ 1,\ 2), & (0,\ 1,\ -\omega^2), & (0,\ 1,\ -\omega), \\ (-\omega^2,\ 0,\ 1), & (-1,\ 0,\ 1), & (-\omega,\ 0,\ 1), \\ (\omega^2,\ 1,\ 1), & (\omega,\ 1,\ 1), & (1,\ 1,\ 1). \end{array}$$

Each of these derived sets has, in addition to the two selected pairs of imaginary points, one real point in common with the set (A). Hence:

(3) There are nine, and only nine, real pencils of cubics with one real and two pairs of imaginary points of inflexion in common with a given real pencil.

These ten sets contain only twenty-eight distinct points:—namely the nine of the set (A) and nineteen others related to the nine by equianharmonic properties. The ten pencils of cubics have sixteen real inflexional axes: the four corresponding to the original set (A),

$$x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 0;$$

and twelve others,

$$\begin{array}{rrr} y - z = 0, & -x + z = 0, & x - y = 0, \\ -x + y + z = 0, & x - y + z = 0, & x + y - z = 0, \\ -2x + y + z = 0, & x - 2y + z = 0, & x + y - 2z = 0, \\ -x + 2y + 2z = 0, & 2x - y + 2z = 0, & 2x + 2y - z = 0. \end{array}$$

These points and lines together with the points and lines found when the two real and a pair of imaginary points were selected, form a complete projective figure considered in the following section.

§ 3. The C_{13} Configuration.

The combination of theorems one and three in the two preceding sections gives the following:

(4) There are twelve, and only twelve, real pencils of cubics with five points of inflexion in common with a given real pencil.

The inflexional sets for these cubics contain in all only forty distinct points, the nine of the original set, the twelve others in the sets having three real and one imaginary pair in common with the original, and the nineteen others in the sets having in common with the original one real and two imaginary pairs. These forty points, ten real and fifteen imaginary pairs, have a notable arrangement on lines.

The thirteen pencils of cubics (the original pencil and the twelve having five points of inflexion in common with it) have nineteen real inflexional axes: the four of the original pencil and the set of three and set of twelve added by the twelve pencils. The nineteen, as determined in the two preceding sections, are arranged in four sets of three pairs in elliptic involution: —namely

$$(1)\quad \left\{\begin{array}{rr} x=0, & -x+2y+2z=0; \\ y+z=0, & -2x+y+z=0; \\ -x+y+z=0, & x+y+z=0; \end{array}\right.$$

$$(2)\quad \left\{\begin{array}{rr} y=0, & 2x-y+2z=0; \\ z+x=0, & x-2y+z=0; \\ x-y+z=0, & x+y+z=0; \end{array}\right.$$

$$(3)\quad \left\{\begin{array}{rr} z=0, & 2x+2y-z=0; \\ x+y=0, & x+y-2z=0; \\ x+y-z=0, & x+y+z=0; \end{array}\right.$$

$$(4)\quad \left\{\begin{array}{rr} y-z=0, & -2x+y+z=0; \\ z-x=0, & x-2y+z=0; \\ x-y=0, & x+y-2z=0; \end{array}\right.$$

with the four pairs of imaginary axes of the original pencil,

$$\begin{array}{rr} \omega x+y+z=0, & \omega^2x+y+z=0; \\ x+\omega y+z=0, & x+\omega^2y+z=0; \\ x+y+\omega z=0, & x+y+\omega^2z=0; \\ x+\omega y+\omega^2z=0, & x+\omega^2y+\omega z=0; \end{array}$$

respectively as double lines. These pencils in involution have for vertices the three real points of the original inflexional set (0, 1, —1), (—1, 0, 1), (1, —1, 0); and the point (1, 1, 1) called in the further discussion the "harmonic point," because it is common to the three real harmonic polars for the original pencil of cubics and is also related to the real inflexional triangle as harmonic point with respect to the line of reals.

Thus as regards the configuration of thirteen pencils of cubics, here denoted by C_{13}, the following theorem may be stated:

(5) The nineteen real inflexional axes of C_{13} are the four of the original pencil and fifteen associates uniquely determined by harmonic properties,

which form four pencils in elliptic involution, with the three real points of inflexion and harmonic point of the original pencil as vertices and the four pairs of imaginary inflexional axes of the original pencil as double lines (Fig. 1).

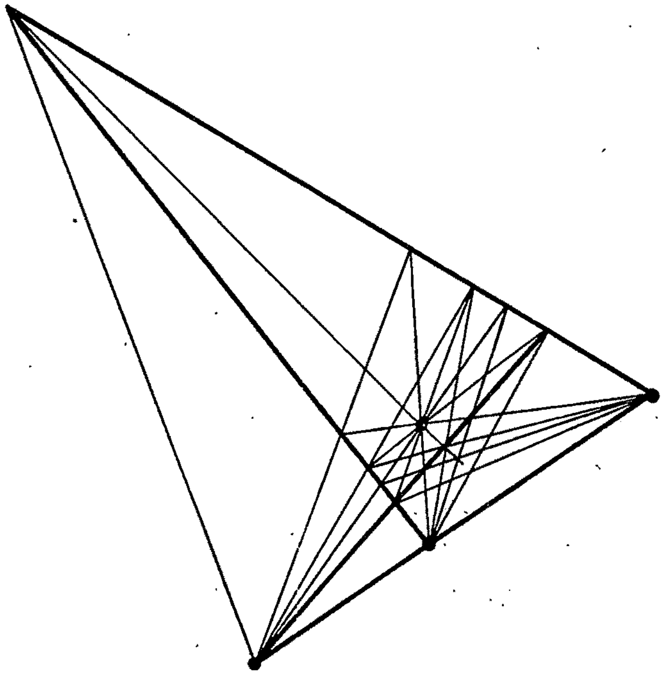

FIG. 1.

The forty points all lie on thirteen of the real inflexional axes, namely the four of the original pencil,

$$x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 0,$$

forming a quadrilateral, the sides of the accompanying diagonal triangle,

$$y + z = 0, \quad z + x = 0, \quad x + y = 0;$$

and the six lines

$$\begin{array}{ccc} y - z = 0, & z - x = 0, & x - y = 0, \\ -x + y + z = 0, & x - y + z = 0, & x + y - z = 0, \end{array}$$

completing a harmonic pencil at each of the six vertices of the quadrilateral. (Fig. 2). Thus the forty points of inflexion of C_{13}, ten real and fifteen imaginary pairs, lie on thirteen real lines which form a complete projective figure.*

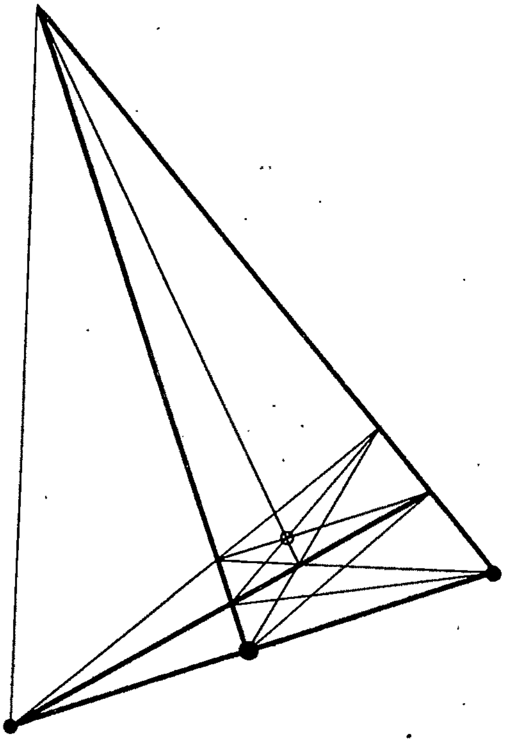

FIG. 2.

The pairs of lines in elliptic involution through the harmonic point (1, 1, 1) are the real harmonic polars,

$$y - z = 0, \quad z - x = 0, \quad x - y = 0,$$

for the original pencil, together with the lines

$$-2x + y + z = 0, \quad x - 2y + z = 0, \quad x + y - 2z = 0,$$

which join the harmonic point to the three real points of inflexion. Thus when three collinear real points and an arbitrary fourth real point are taken respectively as the three real inflexions and harmonic point for a cubic, the three real harmonic polars have the following unique construction.

---

* The figure consists of the special Desargues' configuration 10_3, known as the quadrangle-quadrilateral configuration, and the diagonal triangle common to the associated 4-point and 4-line. The further relation of the C_{13} to this configuration of points and lines are discussed in a paper published in the *Annals of Mathematics*, Second Series, Vol. 26, Nos. 1 and 2, Sept.-Dec., 1924.

> Draw the three lines joining the harmonic point to the three points of inflexion and construct the lines harmonic to each of the three with respect to the other two.

Because of this relation, the three lines joining the harmonic point to the three real inflexions may be called the "harmonic polar associates"; and for the cubics under consideration the intersections of these lines with

$$x = 0, \quad y = 0, \quad z = 0,$$

are

$$\begin{array}{lll} (0,\ 1,\ -1), & (-1,\ 0,\ 1), & (1,\ -1,\ 0), \\ (0,\ 1,\ 2), & (2,\ 0,\ 1), & (1,\ 2,\ 0), \\ (0,\ 2,\ 1), & (1,\ 0,\ 2), & (2,\ 1,\ 0), \end{array}$$

which together with (1, 1, 1) are the ten real points of the determined set of forty. That is:

(6) With respect to a given pencil of cubics, the three real points of inflexion, the harmonic point, and the six other intersections of the harmonic polar associates with the sides of the real inflexional triangle, are the ten real points of inflexion for the C_{13} determined by the pencil. (Fig. 3).

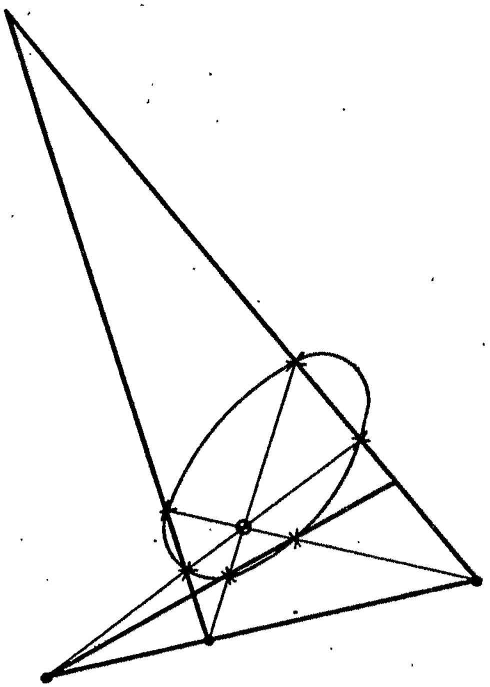

FIG. 3

These ten points are further related. The six

$$(0, 1, 2), \quad (2, 0, 1), \quad (1, 2, 0),$$
$$(0, 2, 1), \quad (1, 0, 2), \quad (2, 1, 0),$$

lie on the conic; and on each of the harmonic polar associates two of these six are harmonic with respect to the harmonic point $(1, 1, 1)$ and one of the three $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$.

The thirty imaginary points of inflexion of a C_{13} (the imaginary points of the group of forty) lie by sixes on the eight imaginary inflexional axes of the original pencil.

II. *Cubics with Three Real Inflexions.*

The results of the discussion of the quadrangle consisting of two real inflexions and an imaginary pair (in which case a third real inflexion is fixed) suggest the consideration of the system of cubics determined by three real inflexions. It is convenient, however, to consider first the more restricted system determined by three real points of inflexion and a fourth real point as harmonic point.

§ 1. Three Real Inflexions and a given Harmonic Point.

It was noted that for a syzygetic pencil of cubics the harmonic point and line of reals are harmonic point and line with respect to the real inflexional triangle. Consider any cubic with $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$ as real points of inflexion and $(1, 1, 1)$ as harmonic point. The sides of a real triangle passing singly through the points of inflexion have equations

$$\alpha x + y + z = 0, \quad x + \beta y + z = 0, \quad x + y + \gamma z = 0,$$

where α, β, γ are any real numbers. The condition that the polar line of $(1, 1, 1)$ with respect to this triangle be $x + y + z = 0$ is $\alpha = \beta = \gamma$. Thus the sides of the real inflexional triangle of the cubic under consideration have equations of the form

$$\alpha x + y + z = 0, \quad x + \alpha y + z = 0, \quad x + y + \alpha z = 0.$$

It is known that for any non-singular cubic there is one pair of imaginary inflexional axes through the harmonic point, and further * that the lines of this pair together with the three harmonic polar associates form an equiharmonic system. Thus the cubic under consideration has as one pair of inflexional axes the lines equiharmonic to

---

* P_1, page 265.

$$-2x+y+z=0, \quad x-2y+z=0, \quad x+y-2z=0;$$

that is,

$$x+\omega y+\omega^2 z=0, \quad x+\omega^2 y+\omega z=0.$$

The intersections of these imaginary lines with the sides of the real inflexional triangle,

$$\begin{aligned} &(\omega-\omega^2,\ a\omega^2-1,\ 1-a\omega), && (\omega^2-\omega,\ a\omega-1,\ 1-a\omega^2); \\ &(1-a\omega,\ \omega-\omega^2,\ a\omega^2-1), && (1-a\omega^2,\ \omega^2-\omega,\ a\omega-1); \\ &(a\omega^2-1,\ 1-a\omega,\ \omega-\omega^2), && (a\omega-1,\ 1-a\omega^2,\ \omega^2-\omega); \end{aligned}$$

for any arbitrary value of a, together with

$$(0,\ 1,\ -1), \quad (-1,\ 0,\ 1), \quad (1,\ -1,\ 0),$$

satisfy the conditions imposed upon the inflexions of a cubic. Consequently the cubic under consideration is any one of the doubly infinite system represented by the equation

$$\begin{aligned} &(x+y+z)\ (x+\omega y+\omega^2 z)\ (x+\omega^2 y+\omega z) \\ &\qquad +\lambda(ax+y+z)\ (x+ay+z)\ (x+y+az)=0, \end{aligned}$$

depending upon two variable parameters which enter one in the first and one in the third degree. To every value of the parameter a there corresponds a unique pencil, hence

(1) All plane cubics with the same three real points of inflexion and a given harmonic point form a doubly infinite system consisting of a single infinity of syzygetic pencils.

As the parameter a varies from zero to infinity the sides of the real inflexional triangle describes three pencils of real lines, intersecting the fixed pair of imaginary inflexional axes in three singly infinite systems of pairs of imaginary points. These points (except in the two special cases considered in the next two paragraphs) are all distinct, since two real lines cannot intersect in an imaginary point. However, these are not all the points on the two imaginary axes, for they are intersected by every one of the double infinity of real lines in the plane (excepting those through their real point of intersection) in a pair of imaginary points, and no two of the intersections coincide for the reason stated. From this follows the theorem:

(2) The imaginary points of inflexion of the doubly infinite system of cubics determined by three real points of inflexion and a given harmonic point, are the intersections of the pair of lines equiharmonic with respect to the harmonic polar associates by the three pencils of lines through the three fixed inflexions.

The three lines

$$ax + y + z = 0,$$
$$x + ay + z = 0,$$
$$x + y + az = 0,$$

form a triangle unless the determinant formed by the coefficients vanishes, that is unless $a = -2$ or 1. If $a = -2$, the three lines intersect in the harmonic point and the curves form a pencil of nodal cubics, that is, the six imaginary points of inflexion coalesce at the harmonic point where the cubics then have a node.* Since the three lines are concurrent for no other value of a, this coalescence can occur at no other point. Hence,

(3) Included in the doubly infinite system of cubics determined by three real points of inflexion and a fixed harmonic point there is precisely one pencil of rational cubics:—namely, a single infinity with a common node at the harmonic point.

If $a = 1$, the three lines are coincident and the curves form a pencil of reducible cubics

$$(x + y + z)\ (x + \omega y + \omega^2 z)\ (x + \omega^2 y + \omega z) + \lambda (x + y + z)^2 = 0,$$

consisting of the line of reals and a pencil of conics with the pair of imaginary axes as tangents and the line of reals as chord of contact, that is, the three pairs of imaginary points of inflexion coincide at the two points of contact.

The vertices of the single infinity of real inflexional triangles

$$(-a - 1, 1, 1),\quad (1, -a - 1, 1)\quad (1, 1, -a - 1),$$

as a varies trace out the lines

$$y - z = 0,\quad z - x = 0,\quad x - y = 0,$$

which are the three real harmonic polars for every curve of the system. Hence

---

* A node on a cubic with three real points of inflexion is necessarily an isolated point.

(4) The locus of the vertices of the real inflexional triangles, for the doubly infinite system of cubics determined by three real points of inflexion and a fixed harmonic point, is the set of three real harmonic polars common to all the curves of the system.

To each of the single infinity of pencils of cubics of this system there corresponds a C_{13}, consisting of the pencil in question and twelve others having five points of inflexion in common with it. Every such C_{13} has in its inflexional group of forty points the three real points of inflexion and harmonic point common to all the curves of the system. The six other real points are the intersections of the sides of the real inflexional triangle by the three harmonic polar associates,

$$-2x+y+z=0,\quad x-2y+z=0,\quad x+y-2z=0,$$

expressible as pairs

$$\begin{array}{ll}(a-1,\ 2a+1,\ -3), & (a-1,\ -3,\ 2a+1);\\ (-3,\ a-1,\ 2a+1), & (2a+1,\ a-1,\ -3);\\ (2a+1,\ -3,\ a-1), & (-3,\ 2a+1,\ a-1).\end{array}$$

As a varies these pairs trace out respectively the three harmonic polar associates. The harmonic point and three fixed inflexions are given by the values of $a(-2$ and $1)$ corresponding to the singular cubics included in the system. This gives the theorem

(5) The locus of the real points of inflexion of the C_{13}, for the doubly infinite system of cubics determined by three real points of inflexion and a fixed harmonic point, is the set of three harmonic polar associates common to all the curves of the system.

As

$$\begin{array}{ll}y-z=0, & -2x+y+z=0;\\ z-x=0, & x-2y+z=0;\\ x-y=0, & x+y-2z=0;\end{array}$$

is a pencil of lines in involution at $(1, 1, 1)$ with

$$x+\omega y+\omega^2 z=0,\quad x+\omega^2 y+\omega z=0$$

as double lines,

(6) The locus of the vertices of the real inflexional triangles and of the real points of inflexion of the C_{13}, for the doubly infinite system of cubics determined by three real points of inflexion and a fixed harmonic point,

is a pencil of six lines in elliptic involution with the harmonic point as vertex and the two imaginary lines through the imaginary points of inflexion as double lines.

§ 2. Three Real Inflexions.

The argument in the preceding section with respect to the system of cubics with

$$(0, 1, -1), \quad (-1, 0, 1), \quad (1, -1, 0)$$

as real points of inflexion and (1, 1, 1) as harmonic point is independent of the choice of coordinates for the harmonic point. Consequently there is a double infinity of such systems with the three inflexions. The removal of the restriction that the curves have a fixed harmonic point gives the system two more degrees of freedom and accounts for the four-fold infinity of cubics in a plane with three common real points of inflexion.

Corresponding to the three pairs of lines forming a pencil in elliptic involution through the harmonic point (1, 1, 1) :—namely

$$\begin{aligned} y - z &= 0, & -2x + y + z &= 0; \\ z - x &= 0, & x - 2y + z &= 0; \\ x - y &= 0, & x + y - 2z &= 0; \end{aligned}$$

with

$$x + \omega y + \omega^2 z = 0, \quad x + \omega^2 y + \omega z = 0$$

as double lines, there are three pairs of points forming a range in elliptic involution on the line of reals $x + y + z = 0$:—namely

$$\begin{aligned} &(0, 1, -1), & &(-2, 1, 1); \\ &(-1, 0, 1), & &(1, -2, 1); \\ &(1, -1, 0), & &(1, 1, -2); \end{aligned}$$

with

$$(1, \omega, \omega^2), \quad (1, \omega^2, \omega);$$

as double points. The removal of the restriction with regard to the harmonic point leaves the range of points invariant; while the totality of harmonic polar associates and real harmonic polars for the four-fold infinity of cubics form six pencils of real lines through the points of the range, and the inflexional axes through the harmonic points form a doubly infinite system of imaginary lines through each of the two double points. Since every imaginary line in a real plane passes through one, and only one, real point, these inflexional axes include all the imaginary lines (in the plane of the cubics) through the two double points of the involution. Consequently the following theorems hold for the quadruply infinite system of cubics in a plane determined by three real points of inflexion:

(1) The imaginary points of inflexion are the intersections of the two doubly infinite systems of imaginary lines through the two points equianharmonic with respect to the three fixed inflexions, by the three pencils of real lines through the three inflexions.

(2) The locus of the vertices of the real inflexional triangles is a double infinity of sets of three lines, one set through each real point of the plane, the lines of each set being the three real harmonic polars common to the curves of a doubly infinite system of the cubics; the whole forming three pencils, one through each of the points harmonic to one of the fixed inflexions with respect to the other two.

(3) The locus of the real points of inflexion of the C_{13} is a double infinity of sets of three lines, one set through each real point of the plane, the lines of each set being three harmonic polar associates common to the curves of a doubly infinite system of the cubics; the whole forming three pencils, one through each of the three fixed inflexions.

(4) The locus of the vertices of the real inflexional triangles and of the real points of inflexion of the C_{13} is a double infinity of sets of six lines, one set through each real point of the plane, each set forming a pencil of three pairs in elliptic involution with the common harmonic point of a doubly infinite system of the cubics as vertex and the imaginary lines through the imaginary points of inflexion of the same doubly infinite system as double lines; the whole forming six complete pencils of lines through the three fixed inflexions and their conjugates in the elliptic involution on the line of reals, having the vertices of the systems of imaginary lines through the imaginary inflexions as double points.

WEST VIRGINIA UNIVERSITY.

A Mathematical Theory of Competition.

BY C. F. ROOS.

1. INTRODUCTION. With reference to the fact that the demand for a commodity often depends not merely on the price at any given time, but also on whether the price is increasing or decreasing at this time,—or, more exactly, assuming that the demand is a linear function both of the price and of the rate of change of price as well,—G. C. Evans has developed a theory of monopoly.* The following is an account of a corresponding theory of competition, which seems to have new elements of mathematical interest. In order to save space the author limits himself, as far as possible, to these latter considerations.

Let us assume there are two producers, each manufacturing subject to the same cost function,

$$q(u) = Au^2 + Bu + C \tag{1}$$

amounts u_1 and u_2, respectively in unit time, and each trying to make his profit Π_i, $i = 1, 2$, a maximum.

If we assume that the demand is a linear function of the price p, and the rate of change of price, dp/dt, and further that as many units are sold as are produced then,

$$y(t) = u_1 + u_2 = ap(t) + h\ \frac{dp}{dt}\ + b \tag{2}$$

where as in Evans' earlier paper † we take $a < 0$, $b > 0$, $A > 0$, $B > 0$, $C > 0$.

In the theory of competition the following two problems are of interest.

PROBLEM (1). Given $p = p_1$ at $t = t_1$ (h being negative) choose u_1 so that Π_1 is a maximum when u_2 is regarded as not subject to variation, and at the same time choose u_2 so that Π_2 is a maximum when u_1 is regarded as not subject to variation.

In this case the restriction on p at $t = t_1$ involves no restriction on u_1 and u_2 since (2) involves an arbitrary constant.

---

* G. C. Evans, "The Dynamics of Monopoly," *Amer. Math. Monthly*, Vol. XXXI, No. 2, Feb., 1924.

† G. C. Evans, "A Simple Theory of Competition," *Amer. Math. Monthly*, Vol. XXIX, No. 10, Nov.-Dec., 1922.

PROBLEM (2). Given $p = p_2$ at $t = t_2$ and $p = p_1$ at $t = t_1$, choose p as a function of t so that Π_1 will be a maximum when the function u_1 alone varies with p and so that Π_2 will be a maximum when the function u_2 alone varies with p.

In this case the variation of u_1 (or u_2) is not arbitrary on account of (2) and the boundary conditions.

In both cases the total profits during the interval of time t_1 to t_2 will be given respectively by,

$$\Pi_1 = \int_{t_1}^{t_2} [pu_1 - q(u_1)]\,dt \tag{3}$$

$$\Pi_2 = \int_{t_1}^{t_2} [pu_2 - q(u_2)]\,dt. \tag{4}$$

2. CONDITIONS FOR MAXIMUM. These problems do not seem to reduce strictly to problems in the calculus of variations but solutions, however, are given below. Let us discuss them in the order already given.

In order to obtain a solution of problem (1) let $\bar{u}_1(t)$, $\bar{u}_2(t)$, and $\bar{p}(t)$ be the values of u_1, u_2, and p giving the desired solution if such values exist. Replace u_1, u_2, and p by,

$$\begin{aligned} u_1 &= \bar{u}_1 + \omega_1\psi_1(t) \\ u_2 &= \bar{u}_2 + \omega_2\psi_2(t) \\ p &= \bar{p} + \omega\theta(t) \end{aligned} \tag{5}$$

where $\psi_1(t)$ and $\psi_2(t)$ are continuous functions of t admitting continuous derivatives in the interval t_1 to t_2 and $\theta(t)$ is a continuous function of t admitting continuous derivatives in this interval and vanishing at t_1. By (2) it is easily seen that $\theta(t)$ satisfies the differential equation $a\omega\theta + h\omega\theta' = \omega_1\psi_1 + \omega_2\psi_2$ and is therefore determined by its value at one point say t_1. It is evident that $\omega_1\psi_1(t)$, $\omega_2\psi_2(t)$, and $\omega\theta(t)$ will be less in absolute value than a fixed quantity ϵ in all the intervals provided $|\omega_1|$ and $|\omega_2|$ are small enough.

Obviously, $u_1 = \bar{u}_1$, $u_2 = \bar{u}_2$, and $p = \bar{p}$ when $\omega_1 = \omega_2 = 0$, because of (2).

Now if we replace u_1, u_2, and p in (3) and (4) by their values as given by (5) we get Π_1 and Π_2 as functions of ω_1 and ω_2; hence the necessary conditions for a solution of our problems are,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \frac{\partial \Pi_2}{\partial \omega_2} = 0 \quad \text{when } \omega_1 = \omega_2 = 0.$$

By the ordinary rules of differentiation applied to (3),

$$(6) \qquad \frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \left[\frac{\partial u_1}{\partial \omega_1} p + u_1 \frac{\partial p}{\partial \omega_1} - q'(u_1) \frac{\partial u_1}{\partial \omega_1} \right] dt.$$

By (5) this expression becomes,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \{ \psi_1(t) [\bar{p} + \omega\theta(t)] + [\bar{u}_1 + \omega_1 \psi_1(t)] \frac{\partial p}{\partial \omega_1}$$
$$- [2A(\bar{u}_1 + \omega_1 \psi_1(t) + B] \psi_1(t) \} dt.$$

From the initial conditions and (2) and (5),

$$\omega\theta(t) = e^{-(a/h)t} \int_{t_1}^{t} \frac{\omega_1 \psi_1(\tau) + \omega_2 \psi_2(\tau)}{h} e^{(a/h)\tau} d\tau, \text{ so that}$$

$$\frac{\partial p}{\partial \omega_1} = \frac{1}{h} e^{-(a/h)t} \int_{t_1}^{t} \psi_1(\tau) e^{(a/h)\tau} d\tau.$$

Substituting this in (6) gives,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \psi_1(t) [\bar{p} + \omega\theta(t) - 2A(\bar{u}_1 + \omega_1 \psi_1(t)) - B$$
$$+ \frac{w_1}{h} \int_{t_1}^{t_2} e^{-(a/h)(t-\tau)} \psi_1(\tau) d\tau] dt + \int_{t_1}^{t_2} dt \int_{t_1}^{t} \frac{\bar{u}_1(t) e^{-(a/h)(t-\tau)} \psi_1(\tau) d\tau}{h}$$

Before we can draw any definite conclusions from the above expression it is necessary to write the iterated integral in a different form by applying Dirichlet's formula and then interchanging t and τ, the parameters of integration. This gives,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} [\bar{p} + \omega\theta(t) - 2A(\bar{u}_1 + \omega_1 \psi_1(t)) - B + \omega_1/h$$
$$\int_{t_1}^{t} e^{-(a/h)(t-\tau)} \psi_1(\tau) d\tau + 1/h \ e^{(a/h)t} \int_{t}^{t_2} \bar{u}_1(\tau) e^{-(a/h)\tau} d\tau] \psi_1(t) dt$$

and for $\omega_1 = \omega_2 = 0$ this becomes

$$\left(\frac{\partial \Pi_1}{\partial \omega_1} \right)_{\omega_1 = \omega_2 = 0} = \int_{t_1}^{t_2} [\bar{p} - 2A\bar{u}_1 - B + 1/h \ e^{(a/h)t} \int_{t}^{t_2} \bar{u}_1(\tau) e^{-(a/h)\tau} d\tau] \psi_1(t) dt = 0.$$

Since $\psi_1(t)$ is arbitrary, we can conclude that the integrand is zero and therefore write as a necessary condition,

$$(7) \qquad p - 2Au_1 - B + 1/h \ e^{(a/h)t} \int_{t}^{t_2} u_1(\tau) e^{-(a/h)\tau} d\tau = 0.$$

In an entirely similar manner we derive from (4)

$$p - 2Au_2 - B + 1/h\ e^{(a/h)t} \int_t^{t_2} u_2(\tau) e^{-(a/h)\tau} d\tau = 0. \tag{8}$$

These equations are proved to be sufficient by showing that $\partial\Pi_1/\partial\omega_1 < 0$ when $\omega_2 = 0$ and $0 < \omega_1 < 1$, and $\partial\Pi_2/\partial\omega_2 < 0$ when $\omega_1 = 0$ and $0 < \omega_2 < 1$, no matter what the functions $\psi_1(t)$ and $\psi_2(t)$ are. The reasoning follows: When (7) is true, (6) becomes

$$\frac{\partial\Pi_1}{\partial\omega_1} = \int_{t_1}^{t_2} \psi_1(t)\left[\omega\theta(t) - 2A\omega_1\psi_1(t) + \omega_1/h \int_{t_1}^{t} e^{-(a/h)(t-\tau)}\psi_1(\tau)d\tau\right]dt.$$

Let $\theta_1(t)$ be the value of θ when $\omega_2 = 0$. Then from (2) and (5)

$$\omega_1\psi_1(t) = a\omega\theta_1(t) + h\omega\theta_1'(t)$$

or solving this differential equation for $\omega\theta_1(t)$ we can write

$$\omega\theta_1(t) = (\omega_1/h)e^{-(a/h)t} \int_{t_1}^{t} e^{+(a/h)\tau}\psi_1(\tau)\,d\tau$$

If we substitute for $\psi_1(t)$, $\omega_1\psi_1(t)$ and $\omega_1/h \int_{t_1}^{t} e^{-(a/h)(t-\tau)}\psi_1(\tau)\,d\tau$ their values as given by the relations above, then

$$\frac{\partial\Pi_1}{\partial\omega_1} = \int_{t_1}^{t_2} \frac{a\omega\theta_1(t) + h\omega\theta_1'(t)}{\omega_1}$$

$$[\omega\theta_1(t) - 2A(a\omega\theta_1(t) + h\omega\theta_1'(t)) + \omega\theta_1(t)]dt$$

$$= 2a\int_{t_1}^{t_2} \frac{\omega^2\theta_1^2(t)}{\omega_1}dt - 2A\int_{t_1}^{t_2} \frac{[a\omega\theta_1(t) + h\omega\theta_1'(t)]^2}{\omega_1}dt$$

$$+ 2h\int_{t_1}^{t_2} \frac{\omega^2\theta_1(t)\theta_1'(t)}{\omega_1}dt$$

Now by an integration performed on the last term of this expression we can write

$$\frac{\partial\Pi_1}{\partial\omega_1} = 2a/\omega_1 \int_{t_1}^{t_2} \omega^2\theta_1^2(t)dt - 2A/\omega_1 \int_{t_1}^{t_2}[a\omega\theta_1(t) + h\omega\theta_1'(t)]^2dt$$

$$+ \left[h\,\omega^2/\omega_1\,\theta_1^2(t)\right]_{t_1}^{t_2}$$

The first two terms of this expression are negative (see (2)), and the term outside the integral sign vanishes at t_1, so that it is negative when h is negative. It is evident that the expression is negative independently of the sign of h if it happens that,

$$\left| 2a/\omega_1 \int_{t_1}^{t_2} \omega^2\theta_1{}^2 dt - 2A/\omega_1 \int_{t_1}^{t_2} [a\omega\theta_1 + h\omega\theta_1{}']^2 dt \right| > \left| h\,\omega^2/\omega_1\,\theta_1{}^2(t_2) \right|$$

but we shall not attempt to discuss this phase of the problem. It is perhaps worth mentioning that if we be given $p = p_2$ at $t = t_2$ instead of $p = p_1$ at $t = t_1$ we can find u_1 as a function of t so that Π_1 during the interval of time t_1 to t_2 will be a maximum and u_2 as a function of t so that Π_2 will be a maximum provided h is positive.

3. Solution of Integral Equations (7) and (8). We wish to find p as an explicit function of t such that it will satisfy equations (7) and (8) and (2), or in other words solve these three equations for the three unknowns u_1, u_2, and p. Adding (7) and (8) and remembering that $u_1 + u_2 = ap + b + hp'$ we get,

$$(9) \qquad he^{-(a/h)t}\,[2p - 2A(ap + b + hp') - 2B] + \int_t^{t_2} (ap + b + hp')\,e^{-(a/h)\tau} d\tau = 0.$$

Before proceeding further let us notice that this expression conditions the value of p at $t = t_2$, i. e. p must satisfy the equation,

$$(10) \qquad 2p - 2A(ap + b + hp') - 2B = 0 \quad \text{when } t = t_2.$$

The integral equation (9) can readily be reduced to a second order differential equation with constant coefficients by a differentiation with respect to t. We get on collecting terms,

$$(11) \qquad -2Ah^2p'' + hp' - a(3 - 2Aa)p = b - 2Aab - 2Ba.$$

The general solution of this equation is,

$$(12) \qquad p = \frac{b - 2Aab - 2Ba}{-a(3 - 2Aa)} + C_1 e^{m_1 t} + C_2 e^{m_2 t} \qquad \text{where}$$

$$m_2 = \frac{1 + \sqrt{1 - 8Aa(3 - 2Aa)}}{4Ah} \text{ and } m_2 = \frac{1 - \sqrt{1 - 8Aa(3 - 2Aa)}}{4Ah}$$

and C_1 and C_2 are determined from the initial condition $p = p_1$ at $t = t_1$, and the condition (10). In fact the determinant of the linear equations in C_1 and $C_2 \neq 0$. There always exists therefore a unique solution of the problem proposed.

Since u_1 and u_2 are solutions of the same integral equation [*see* (7) *and* (8)] *we can conclude that $u_1 = u_2$ and proceed to determine the amount of goods produced by means of* (2).

4. SOLUTION OF PROBLEM 2. Given $p = p_1$ at $t = t_1$, and $p = p_2$ at $t = t_2$ we wish to find u_1 as a function of t so that Π_1 is a maximum, and u_2 as a function of t so that Π_2 is a maximum no matter what the variation of p.

Let $\psi_1(t)$, $\psi_2(t)$, and $\theta(t)$ be continuous functions of t with their derivatives and write

$$\begin{aligned} u_1 &= \bar{u}_1 + \omega_1\psi_1(t) \\ u_2 &= \bar{u}_2 + \omega_2\psi_2(t) \\ p &= \bar{p} + \omega\theta(t) \end{aligned}$$

where $\bar{u}_1$, $\bar{u}_2$, and $\bar{p}$ are the values of u_1, u_2, and p giving the desired solution. The three families $\omega_1\psi_1$, $\omega_2\psi_2$, $\omega\theta$ are therefore arbitrary except that they are always related by $\omega_1\psi_1 + \omega_2\psi_2 = a\omega\theta + h\omega\theta'$. Let us now impose the further condition that $\theta(t)$ vanish at $t = t_1$ and at $t = t_2$.

As before we get $u_1 = \bar{u}_1$, $u_2 = \bar{u}_2$ and $p = \bar{p}$ when $\omega_1 = \omega_2 = 0$. The necessary conditions for Π_1 to be a maximum as a function of u_1 and Π_2 a maximum as a function of u_2 are $\partial\Pi_1/\partial\omega_1 = \partial\Pi_2/\partial\omega_2 = 0$ when $\omega_1 = \omega_2 = 0$.

For the determination of u_1, the parameter ω_2 will be zero, and therefore the work is simplified by considering the family $\omega\theta_1(t)$ for which $\omega_2 = 0$. In fact, we can write $\omega_1\psi_1(t) = a\omega\theta_1(t) + h\omega\theta_1'(t)$.

As in article (2) we obtain,

$$\frac{\partial\Pi_1}{\partial\omega_1} = \int_{t_1}^{t_2} \left(\frac{\partial u_1}{\partial\omega_1} p + u_1 \frac{\partial p}{\partial\omega_1} - q'(u_1)\frac{\partial u_1}{\partial\omega_1}\right) dt$$

where $\dfrac{\partial p}{\partial\omega_1} = \dfrac{1}{h} e^{-(a/h)t} \displaystyle\int_{t_1}^{t} \psi_1(\tau)\, e^{(a/h)\tau} d\tau = \omega/\omega_1\, \theta_1(t)$.

By means of the relation $\omega_1\psi_1 = a\omega\theta_1 + h\omega\theta_1'$ this expression becomes,

$$\text{(A)} \quad \begin{aligned} \frac{\partial\Pi_1}{\partial\omega_1} &= \int_{t_1}^{t_2} \left\{ \frac{a\omega\theta_1 + h\omega\theta_1'}{\omega_1}(\bar{p} + \omega\theta_1) + \bar{u}_1\, \omega/\omega_1\, \theta_1 \right. \\ &\qquad \left. - [2A(\bar{u}_1 + a\omega\theta_1 + h\omega\theta_1') + B]\, \frac{a\omega\theta_1 + h\omega\theta_1'}{\omega_1} \right\} dt \\ &= \int_{t_1}^{t_2} \{ a\bar{p} + a\omega\theta_1 + h\omega\theta_1' + \bar{u}_1 - 2Aa\bar{u}_1 - 2Aa^2\omega\theta_1 - 2Aah\omega\theta_1' \\ &\qquad - Ba - ah\omega\theta_1' \}\, \omega/\omega_1\, \theta_1\, dt \\ &\quad + \int_{t_1}^{t_2} (hp\theta_1' - 2Ah\bar{u}_1\theta_1' - 2Ah^2\omega\theta_1'^2 - hB\theta_1')\, \omega/\omega_1\, dt. \end{aligned}$$

By integrating by parts the first three terms under the second integral sign and remembering $\theta(t_1) = \theta(t_2) = 0$ this can be written as,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} [a\bar{p} + a\omega\theta_1 + h\omega\theta_1' + \bar{u}_1 - 2Aa\bar{u}_1 - 2Aa^2\omega\theta_1 - 2Aah\omega\theta_1' \\ - Ba - ah\omega\theta_1' - h\bar{p}' + 2Ah\bar{u}_1' + 2Ah^2\omega\theta''] \,\omega/\omega_1\, \theta_1 \, dt$$

and for $\omega_1 = \omega_2 = 0$ this becomes,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} [-h\bar{p}' + a\bar{p} + (1 - 2Aa)\bar{u}_1 - Ba + 2Ah\bar{u}_1']\theta_1(t) \,\omega/\omega_1 \, dt = 0.$$

If we let the value $\omega_1 = 1$ correspond to the value $\omega = 1$, i. e., let $\psi_1(t) = u(t) - \bar{u}(t)$ when $\theta_1(t) = p(t) - \bar{p}(t)$, the relation between $\omega_1\psi_1(t)$ and $\omega\theta_1(t)$ yields the identity $\omega = \omega_1$. But $\theta_1(t)$ is arbitrary except that it vanishes at t_1 and t_2. Hence we can conclude that the above integrand is zero and write

$$-h\bar{p}' + a\bar{p} + (1 - 2Aa)\bar{u}_1 - Ba + 2Ah\bar{u}_1' \equiv 0. \tag{13}$$

In an entirely similar manner we obtain from the condition $\partial \Pi_2/\partial \omega_2 = 0$

$$-h\bar{p}' + a\bar{p} + (1 - 2Aa)\bar{u}_2 - Ba + 2Ah\bar{u}_2' = 0. \tag{14}$$

It happens that these conditions are also sufficient, for when (13) is true the expression (A) for $\partial \Pi_1/\partial \omega_1$ becomes

$$\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} [(a\omega\theta_1 + h\omega\theta_1') \,\omega/\omega_1\, \theta_1 + (a\omega\theta_1 + h\omega\theta_1') \,\omega/\omega_1\, \theta_1 \\ - 2A(a\omega\theta_1 + h\omega\theta_1') \frac{(a\omega\theta_1 + h\omega\theta_1')}{\omega_1} \\ + Bh \,\omega/\omega_1\, \theta_1'] \, dt.$$

If we integrate the terms in θ_1' and remember that θ_1 vanishes at t_1 and at t_2, this can be written as,

$$\frac{\partial \Pi_1}{\partial \omega_1} = \frac{2a}{\omega_1} \int_{t_1}^{t_2} \omega^2\theta_1^2 dt - 2A \int_{t_1}^{t_2} (a\omega\theta_1 + h\omega\theta_1')^2 \, dt$$

and for $\omega_2 = 0$, $0 < \omega_1 < 1$ this is evidently negatived by the inequalities (2).

The conditions (13) and (14) and equation (2) yield the solution of our problem. If we add (13) and (14) we obtain on dropping dashes

$$2ap - 2hp' + (1 - 2Aa)(u_1 + u_2) - 2Ba + 2Ah(u_1' + u_2') = 0$$

and by (2) this becomes

$$-2Ah^2p'' + hp' - a(3 - 2Aa)p = b - 2Aab - 2Ba. \tag{14A}$$

This differential equation is the same as (11) but the constants in the solution are determined differently. The general solution is,

$$p = p_0 + C_1 e^{m_1 t} + C_2 e^{m_2 t}, \tag{15}$$

where m_1 and m_2 have the values previously given, and

$$p_0 = \frac{b - 2Aab - 2Ba}{-a(3 - 2Aa)}$$

For simplicity suppose $t_1 = 0$, then if we let $r_1 = p_1 - p_0$ and $r_2 = p_2 - p_0$, substitution in (15) gives us

$$C_1 = \frac{r_1 - r_2\, e^{-m_2 t_2}}{1 - e^{(m_1 - m_2)t_2}} \quad ; \qquad C_2 = \frac{r_1 - r_2 e^{-m_1 t_2}}{1 - e^{-(m_1 - m_2)t_2}}$$

We can therefore say that a unique solution of the problem proposed always exists.

5. Discussion of Solution of Problem 2. It is interesting to note that although (15) is a solution of a differential equation containing a term in p' while Evans' corresponding result for monopoly is a solution of a differential not involving p' similar conclusions regarding the solution can be made.

A particular solution of (14A) is the constant $p = p_0$, which is the Cournot competition price obtained when the equation of demand does not involve the rate of change of price.*

If we choose the end values such that $p = p_0$ etc., it continues to be a solution of our problem. Also, as in the case of monopoly, no solution not identically equal to p_0 can take on the value p_0 more than once; in fact, the real non-negative value of $t \leqq t_2$ for which $p = p_0$ is given by

$$e^{(m_1 - m_2)t} = -\frac{C_2}{C_1}$$

The derivative dp/dt will be zero for the single real value of t in the interval $0 \leqq t \leqq t_2$ for which

$$e^{(m_1 - m_2)t} = -\frac{m_2 C_2}{m_1 C_1}$$

We can show by using the values of C_1 and C_2 in terms of r_1 and r_2 that, if r_1 and r_2 have opposite signs the graph of price against time cuts the line $p = p_0$ once and has no horizontal tangent in the interval, the price continuously increasing or decreasing as the case may be, from p_1 to p_2; if r_1 and r_2

---

* G. C. Evans already cited.

have the same sign the graph fails to cross the line $p = p_0$ at all and has one and only one horizontal tangent provided the interval of time is large enough. As is seen by direct computation we must have

$$\frac{m_1 - m_2}{m_1 e^{-m_2 t} - m_2 e^{-m_1 t}} \leq r_2/r_1 \leq \frac{m_1 e^{m_2 t} - m_2 e^{m_1 t}}{m_1 - m_2},$$

otherwise there is no horizontal tangent. Since the second derivative does not vanish where dp/dt does we can conclude that p has a maximum in the interval if dp/dt is positive at $t = 0$, and a minimum if dp/dt is negative at $t = 0$. This slope dp/dt has the same algebraic sign as the quantity

$$r_2 - r_1 \frac{m_1 e^{m_2 t_2} - m_2 e^{m_1 t_2}}{m_1 - m_2}.$$

A comparison of this quantity with the previous inequality shows that when r_1 and r_2 are both positive $p(t)$ has only a minimum and when r_1 and r_2 are both negative it has only a maximum. All of the above results are independent of the sign of h.

Since (13) and (14) are necessary and sufficient conditions for the solution of the problem, u_1 and u_2 are not necessarily identical, as functions of t, but may have initial values conditioned merely by the fact that their sum is given by (2).

6. Extension to n Producers. It is instructive to investigate the phenomenon of competition when n producers are involved, inasmuch as we can derive a formula which includes (15) as well as the monopoly formula of Evans.

Altho each of the problems already discussed can be extended to the case of n producers I shall only give the extension of problem (2).

If we let $u_1, u_2, \cdots, u_n$ represent the respective amounts produced by each competitor then,

$$u_1 + u_2 + \cdots + u_n = ap + b + hp'. \tag{2$'$}$$

If $\Pi_1, \Pi_2, \cdots, \Pi_n$ denote the respective profits we have,

$$\Pi_i = \int_{t_1}^{t_2} [pu_i - q(u_i)]\, dt, \qquad i = 1, 2, 3, \cdots, n.$$

where each competitor assumes that the production of the other or others is independent of his and each tries to make his profit a maximum.

Let $\psi_1(t), \psi_2(t), \cdots, \psi_n(t)$ and $\theta(t)$ be continuous functions of t with their derivatives and write,

$$u_i = \bar{u}_i + \omega_i \psi_i(t) \quad \text{where } i = 1, 2, 3, \cdots, n$$
$$p = \bar{p} + \omega\theta(t)$$

where $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$, and $\bar{p}$ are the values of $u_1, u_2, \cdots, u_n, p$ when p is chosen as a function of t so that Π_i will be a maximum when the function u_i alone varies with p. Because of (2)′ the $n + 1$ families $\omega_1\psi_1, \omega_2\psi_2, \cdots, \omega_n\psi_n, \omega\theta(t)$ are always related by

$$\omega_1\psi_1 + \omega_2\psi_2 + \cdots + \omega_n\psi_n = a\omega\theta + h\omega\theta'.$$

The necessary conditions for Π_i to be a maximum when u_i alone varies with p are therefore $\frac{\partial \Pi_1}{\partial \omega_2} = \frac{\partial \Pi_2}{\partial \omega_2} = \cdots = \frac{\partial \Pi_n}{\partial \omega_n} = 0$ when $\omega_1 = \omega_2 = \cdots \omega_n = 0$.

By an analysis similar to that of article (3) we obtain the n necessary and sufficient conditions,

$$-hp' + ap + (1 - 2Aa)u_i - Ba + 2Ahu_i' = 0, \quad i = 1, 2, 3, \cdots, n.$$

If we add these n equations and remember that $u_1 + u_2 \cdots + u_n = ap + b + hp'$ we obtain the second order differential equation,

$$-2Ahp'' + (n - 1)hp' - a(n + 1 - 2Aa)p = b - 2Aab - nBa.$$

The general solution of this differential equation is,

$$(15)' \qquad p = p_0 + C_1'e^{m'_1t} + C_2'e^{m'_2t} \qquad \text{where}$$

$$p_0 = \frac{b - 2Aab - nBa}{-a(n + 1 - 2Aa)},$$

$$m_1' = \frac{(n - 1) + \sqrt{(n - 1)^2 - 8Aa(n + 1 - 2Aa)}}{4Ah},$$

$$m' = \frac{(n - 1) - \sqrt{(n - 1)^2 - 8Aa(n + 1 - 2Aa)}}{4Ah}$$

and C_1' and C_2' are determined as before in terms of r_1 and r_2.

If $n = 2$ our expression (15)′ reduces to the formula discussed for two competitors, and again if $n = 1$ we get the monopoly price already referred to. Again if $n = \infty$, we find that $C_1 \to 0$, $C_2 \to r$, and $p \to B$ for all $t \neq t$, —that is, the larger the number of competitors the more rapid is the adjustment to new price levels.

7. APPLICATION OF INTEGRAL EQUATIONS. In the present paper we have assumed that the demand is a linear function of the price and the rate of change of price. Another likely hypothesis is to consider the demand as depending not only on the present price but on all previous prices as well. This new hypothesis will lead into the theory of integral equations. Such an assumption is of the form,

$$y(t) = ap(t) + b + \int_{-\infty}^{t} \Phi(t - \tau)p(\tau)\,d\tau. \tag{16}$$

where we assume $p(-\infty)$ is finite and $\Phi(t-\tau)$ is negligible if $t-\tau$ is large and negative. This is the usual restriction in treating the phenomena of hysteresis. Consider the problem in which the interval of time extends from $t = -\infty$ to $t = t_2$ and each Π_i is to be a maximum considered as a functional of the corresponding u_i. We have then,

$$\Pi_i = \int_{-\infty}^{t_2} [pu_i - q(u_i)]\,dt.$$

Using the methods of this paper as given in article (2). we get,

$$\begin{aligned} \omega\theta(t) &= \frac{\omega_1\psi_1 + \omega_2\psi_2}{a} - \int_{\infty}^{t} \omega\theta(t)\ \frac{\Phi(t-\tau)}{a}\ d\tau \\ &= \frac{\omega_1\psi_1 + \omega_2\psi_2}{a} + \int_{-\infty}^{t} \frac{\omega_1\psi_1(\tau) + \omega_2\psi_2(\tau)}{a}\ \phi(t-\tau)\,d\tau \end{aligned}$$

where $\phi(t-\tau)$ is the resolving kernel of $\dfrac{\Phi(t-\tau)}{a}$.

Then $$\frac{\partial p}{\partial \omega_1} = \frac{\psi_1(t)}{a} + \int_{-\infty}^{t} \phi(t-\tau)\ \frac{\psi_1(\tau)}{a}\ d\tau$$

so that after applying Dirichlet's formula, we get,

$$\begin{aligned} \frac{\partial \Pi_1}{\partial \omega_1} = \int_{-\infty}^{t_2} \psi_1(t)\Big[p + \omega\theta(t) + \frac{u_1}{a} + \frac{\omega_1\psi_1}{a} - 2A(u_1 + \omega_1\psi_1) - B \\ + \frac{\omega_1}{a} \int_{-\infty}^{t} \phi(t-\tau)\psi_1(\tau)\,d\tau + \int_{t}^{t_2} \frac{u_1(\tau)}{a}\,\phi(\tau - t)\,d\tau\Big]\,dt \end{aligned}$$

and for $\omega_1 = \omega_2 = 0$ this becomes

$$\left(\frac{\partial \Pi_1}{\partial \omega_1}\right)_{\omega_1=\omega_2=0} = \int_{-\infty}^{t_2} \Big[p + u_1/a - 2Au_1 - B + \int_{t}^{t_2} \frac{u_1(\tau)}{a}\,\phi(\tau - t)\,d\tau\Big]\psi_1(t)\,dt.$$

In the same manner as before we have as necessary conditions for a solution of the problem proposed,

$$p+u_1/a-2Au_1-B+1/a\int_t^{t_2}u_1(\tau)\phi(\tau-t)\,d\tau=0,$$

$$p+u_2/a-2Au_2-B+1/a\int_t^{t_2}u_2(\tau)\phi(\tau-t)\,d\tau=0.$$

Adding these two equations gives

$$2p+y(t)/a-2Ay(t)-B+1/a\int_t^{t_2}y(\tau)\phi(\tau-t)\,d\tau=0. \tag{17}$$

Now from (16),

$$\begin{aligned}p&=\frac{y-b}{a}-\int_{-\infty}^{t}\frac{\Phi(t-\tau)}{a}p(\tau)\,d\tau.\\&=\frac{y-b}{a}+\int_{-\infty}^{t}\frac{y(\tau)-b}{a}\phi(t-\tau)\,d\tau.\end{aligned}$$

We can therefore write (17) as

$$\begin{aligned}y(t)={}&\frac{B}{3-2Aa}+\frac{2B}{3-2Aa}\int_{-\infty}^{t}\phi(t-\tau)\,d\tau-\frac{2}{3-2Aa}\\&\int_{-\infty}^{t}\phi(t-\tau)y(\tau)\,d\tau-\frac{1}{3-2Aa}\int_{t}^{t_2}\phi(t-\tau)y(\tau)\,d\tau\end{aligned}$$

If we define:

$$f(t)=\frac{2b}{3-2Aa}\int_{-\infty}^{t}\phi(t-\tau)\,d\tau+\frac{B}{3-2Aa}$$

$$\begin{aligned}K(t\tau)&=-2\phi(t-\tau)\ \text{ when } \tau<t \ \text{ and}\\&=-\phi(\tau-t)\ \text{ when } \tau>t \ \text{ and } \lambda=\frac{1}{3-2Aa}\\&=0\ \text{ when } \tau=t\end{aligned}$$

we can write this Fredholm Integral Equation as,

$$\begin{aligned}y(t)&=f(t)+\lambda\int_{-\infty}^{t_2}K(t,\tau)y(\tau)\,d\tau.\\&=f(t)-\lambda\int_{-\infty}^{t_2}k(t,\tau)f(\tau)\,d\tau,\end{aligned}\tag{18}$$

where $k(t,\tau)$ is the resolving kernel of $K(t,\tau)$ unless λ happens to be a characteristic value. It is perhaps worth while to mention that λ is a positive fraction less than one (see inequalities given in (2).

Returning to the earlier problems, if we add (7) and (8) [the necessary conditions for a solution of problem 1, hypothesis (2)] and substitute for p

its value obtained as a solution of the differential equation (2), the following integral equation for the determination of y results.

$$y(t) = f(t) + \int_{t_2}^{t} K(t - \tau) y(\tau)\, d\tau, \quad \text{where}$$

$$f(t) = -\frac{b + aB}{aA} + \frac{ap_2 + aB}{aA}\, e^{-(a/h)(t-t_2)} \quad \text{and}$$

$$K(t - \tau) = \frac{2e^{-(a/h)(t-\tau)} - e^{(a/h)(t-\tau)}}{2Ah}$$

and p_2 is the value of p at $t = t_2$, to be determined later. This equation can be solved without appealing to the results already obtained in the discussion of problem 1.

In fact, since $K(t - \tau)$ satisfies a simple second order differential equation we can obtain the revolving kernel from the fundamental reciprocal formula. Differentiating twice and combining by $K''(v) - a^2/h^2\, K(v) = 0$ gives a second order differential equation whose solution is,

$$k(t - \tau) = g_1 e^{m_1(t-\tau)} + g_2 e^{m_2(t-\tau)}$$

where m_1 and m_2 are the quantities previously called m_1 and m_2, and

$$g_1 = \frac{2Ahm_2 + 6Aa - 1}{4A^2h^2(m_1 - m_2)}\,; \qquad g_2 = -\frac{2Ahm_1 + 6Aa - 1}{4A^2h^2(m_1 - m_2)}$$

We can therefore write the solution of (19) as

$$y(t) = f(t) - \int_{t_2}^{t} k(t - \tau) f(\tau)\, d\tau \tag{20}$$

where the value of p_2 is determined as follows:

From the initial conditions and the linear differential equation (2) we obtain

$$p_2 = p_1\, e^{-(a/h)(t_2-t_1)} + (1/h)\, e^{-(a/h)t_2} \int_{t_1}^{t_2} e^{(a/h)\tau}\, y(\tau)\, d\tau$$

and by (20) this becomes

$$p_2 = p_1\, e^{-(a/h)(t_2-t_1)} + (1/h)\, e^{-(a/h)t_2} \int_{t_1}^{t_2} e^{(a/h)\tau} f(\tau)\, d\tau - 1/h\, e^{-(a/h)t_2}$$

$$\int_{t_1}^{t_2} e^{(a/h)\tau}\, d\tau \int_{t_2}^{\tau} k(\tau - x) f(x)\, dx$$

an implicit equation for the determination of p_2.

The Rice Institute,
Houston, Texas.

Imprimitive Substitution Groups.

By G. A. Miller.

---

§ I. *Definitions and generalities.*

A transitive group G of degree n is said to be imprimitive whenever the letters of G can be divided into distinct sets or distinct systems which are transformed as units by all the substitutions of G. Each such set or system is supposed to be composed of more than one letter. While the notion of imprimitivity is one of the oldest in the theory of substitution groups, appearing in the works of P. Ruffini and E. Gallois, comparatively little progress has been made as regards the determination of the possible number of different systems of imprimitivity that may appear in the imprimitive groups of a given degree n. In the particular case when such a group is regular it is known that there is a (I, I) correspondence between the possible systems of imprimitivity and the non-identity subgroups of the group. In particular, the regular abelian group of order 32 and of type (I, I, I, I, I) has as many as 372 different systems of imprimitivity.

In view of the fact that the notion of imprimitivity is so fundamental it is not surprising to find that a large number of equivalent definitions of the term imprimitive group can be given. Among these are the following: A necessary and sufficient condition that a substitution group is imprimitive is that all the substitutions of G which transform a given letter into another given letter transform among themselves a certain subset of the letters of G. This subset does not necessarily constitute a system of imprimitivity of G. Another necessary and sufficient condition that G is imprimitive is that for any fixed letter of G it is possible to find another letter such that the totality of the substitutions of G which transform the former into the latter does not generate G. This is equivalent to saying that this totality generates an intransitive group. It may be noted that the letters of the transitive constituent of this intransitive group which include the given fixed letter constitute a system of imprimitivity of G but that this is not necessarily true as regards the letters of the other transitive constituents of this intransitive group. The degree of each of these other transitive constituents is divisible by the degree of the former constituent, and whenever this quotient is unity the letters of the corresponding constituent constitute also a system of imprimitivity of G.

The definitions of imprimitivity which were noted in the preceding paragraph are based directly on the representation of the group as a substitution group. It is interesting to note here also definitions which are based upon abstract properties of G. It is well known that a necessary and sufficient condition that G can be represented as a transitive substitution group on n letters is that it involves a subgroup G_1 of index n which is not invariant under G and does not contain any invariant subgroup of G besides the identity. If we write the augmented co-sets of G with respect to G_1 in the usual way, as follows:

$$G = G_1 + s_2G_1 + s_3G_1 + \cdots + s_nG_1$$

it is well known that a necessary and sufficient condition that the corresponding transitive group of degree n is primitive is that G_1 is a maximal subgroup of G. This is equivalent to saying that a necessary and sufficient condition that this substitution group is imprimitive is that there is at least one co-set $s_\alpha G_1$, where α has one of the values $2, 3, \cdots, n$, which does not generate G. In what follows it will be proved that *a necessary and sufficient condition that none of the co-sets* $s_\alpha G$ *generates* G *is that every pair of letters of this substitution group appears in a possible system of imprimitivity.* Hence we have the following three cases: A necessary and sufficient condition that this substitution group is primitive is that every co-set $s_\alpha G_1$ generates G; a necessary and sufficient condition that it is imprimitive is that at least one of these co-sets does not generate G; a necessary and sufficient condition that every pair of its letters appears in some possible system of imprimitivity is that none of these co-sets generates G.

To prove the italicized theorem of the preceding paragraph it should first be noted that the group generated by $s_\alpha G_1$, where α has any one of the values $2, 3, \cdots, n$, includes G_1. This group is evidently of degree n when G is represented as a transitive group with respect to G_1, as will be assumed in what follows. In fact, G_1 and any substitution of G which is not contained in G_1 obviously generates a group of degree n. When the group generated by $s_\alpha G_1$ is transitive it must therefore coincide with G. When it is intransitive it must involve all the substitutions of G which transform a letter omitted by G_1 into the letters of its transitive constituents which involves this omitted letter. Every one of the remaining substitutions of G must therefore transform all the letters of this transitive constituent into a set of letters which has no letter in common with this constituent. That is, the letters of this constituent constitute a system of imprimitivity of G. Moreover, they constitute the smallest system of imprimitivity of G which involves this omitted letter of G_1 and the letter by which it is replaced in s_α, as a direct consequence of the definition of a system of imprimitivity.

Since every right co-set which involves a fixed substitution generates the same group as the left co-set which involves this substitution and since the former co-sets are composed of all the substitutions of G which transform a letter omitted by G_1 into a fixed letter, it results that this omitted letter appears in a system of imprimitivity containing an arbirarily other letter of G whenever no one of the possible co-sets with respect to G_1 generates G, and vice versa. Hence the theorem under consideration has been established. It also results from the above that to every proper subgroup generated by a co-set, or by more than one co-set with respect to G_1, there corresponds a system of imprimitivity of G which involves the letter omitted by G_1 and to every such system there corresponds such a subgroup. That is, there is a (I, I) correspondence between the different possible systems of imprimitivity of G and the different subgroups generated by one co-set, or by more than one co-set with respect to G_1. A necessary and sufficient condition that G is imprimitive is that some two of the n conjugates of G_1 under G generate a proper subgroup of G.

§ 2. *Number of systems of imprimitivity.*

We shall first consider the different possible systems of imprimitivity of the transitive group G which are such that each system is composed of two letters. When G is non-regular the subgroup G_1 composed of all the substitutions of G which omit a given letter a_1 must omit all the letters of the system including a_1 in the various possible set of systems which are such that each system is composed of two letters, since the substitutions which omit one letter of such a system must also omit the other letter of the system. If the degree of G_1 is $n - n_1$ it is known that G_1 is invariant under a subgroup of G whose order is n_1 times the order of G_1 and which involves a regular constituent of order n_1. As the conjugates of the substitutions of this constituent represent the substitutions which involve the letters of G and are invariant under G it results directly that there is a (I, I) correspondence between the substitutions of order 2 in this regular constituent and the different systems of imprimitivity of G which are such that each system is composed of two letters.

Since n_1 is a division of n it results from the preceding paragraph that the number of the different systems of imprimitivity of G such that each system is composed of two letters is equal to the number of operators of order 2 in a group whose order divides n. Moreover, if n_1 is an arbitrary divisor of n the direct product of n/n_1 regular groups of order n_1 extended by substitutions which transform the transitive constituents of this direct product according to a regular group obviously contains exactly as many systems of imprimitivity, such that each system is composed of two letters, as there are substitutions of order 2 in the given regular group of order n_1. This establishes the following theorem: *It is possible to construct an imprimitive group of degree n which has exactly as many different systems of imprimitivity as there are operators of order 2 in an arbitrary group whose order is an arbitrary divisor of n, including n; and an imprimitive group of degree n cannot have any other number of such systems of imprimitivity.* In particular, if an imprimitive group of degree n has at least one set of systems of imprimitivity such that each system is composed of 2 letters the number of such systems is always odd, and is equal to the number of substitutions of order 2 on the letters of this imprimitive group which are commutative with every one of its substitutions.

The theory of the number of possible systems of imprimitivity such that each system involves a prime number p of letters is much more difficult when p is odd than when p is 2. To illustrate this fact it may be noted that when p is odd it is possible to construct a group of degree p^2 and of order $2p^2$ which has exactly two systems of imprimitivity such that each system involves exactly p letters. To construct such a group it is only necessary to extend the regular non-cyclic group of order p^2 by means of a substitution of order 2 which is commutative with one of the substitutions of order p in this regular group and transforms another of these substitutions into its inverse. The group of order $2p^2$ thus obtained contains 2 and only 2 invariant subgroups of order p while its remaining $p-1$ subgroups of this order are conjugate in pairs. This group obviously contains no systems of imprimitivity besides the two whose systems are composed of p letters each.

If the order of G is p^m each of its systems of imprimitivity such that a system is composed of p letters corresponds to a substitution of order p which is invariant under G. Hence the total number of such systems is found in a group containing only substitutions on the letters of G and whose degree is equal to the degree of G. It therefore results directly that *when p is an odd prime number and G is of order p^m the number of its systems of imprimitivity such that each system involves p letters is equal to the number of the subgroups of order p in a group whose order is a divisor of the degree of G.* In particular, the number of these systems is always of the form $1+kp$. If d represents any divisor of the degree of G, including this degree but excluding the identity, then it is possible to construct an imprimitive group whose degree is equal to the degree of G such that this imprimitive group contains exactly as many systems of imprimitivity in which each system is composed of p letters as there are subgroups of order p in an arbitrary group of order d. Such a group can obviously be constructed by the method used above when p was supposed to be equal to 2.

Every transitive group of order p^m contains systems of imprimitivity such that each system contains a number of letters which is an arbitrary proper divisor of the degree, excluding the identity, since such a group contains an invariant subgroup whose order is an arbitrary divisor of the order of the group. It is interesting to note that every transitive constituent of a Sylow subgroup of a symmetric group has one and only one set of systems of imprimitivity for every such divisor of its degree. If such a Sylow subgroup is transitive then it will also contain one and only one set of systems of imprimitivity for every such divisor of its degree. Hence the following theorem: *Every transitive constituent of a Sylow subgroup of the symmetric group of degree n has one and only one set of systems of imprimitivity such that the number of letters in a system is* p, $p^2, \cdots, p^{a-1}$, p^a being the degree of this transitive constituent and a being greater than unity. In other words, a transitive constituent of a Sylow subgroup of the symmetric group has the smallest possible number of different systems of imprimitivity for a prime power transitive group of this degree.

Suppose that G contains a transitive subgroup whose degree n_1 is less than the degree of G. It is not difficult to see that G contains either no set of systems of imprimitivity such that the number of letters in one system is n_1 or it contains only one such set. In the latter case a system of imprimitivity is composed of the n_1 letters of the subgroup in question. The italicized theorem of the preceding paragraph is obviously a special case of the general theorem just noted. In particular, it results from this theorem that when G involves at least two transitive subgroups of degree n_1 which have some common letters but not all letters in common then it cannot involve systems of imprimitivity such that each system involves exactly n_1 letters. It may also be noted that every non-regular prime power transitive group of degree p^2 contains one and only one set of systems of imprimitivity; viz., the one corresponding to the systems of intransitivity of the invariant subgroup of order p.

UNIVERSITY OF ILLINOIS.

Classification of Monoidal Involutions Having a Fixed Tangent Cone.

By Marian M. Torrey.

I. *Introduction.*

1. If a (1, 2) algebraic correspondence is established between the points of two spaces (x') and (x), such that a point P_1 of (x) uniquely determines a point P' of (x'), but the point P' determines two points of (x), P_1 and a second point P_2, P_1 and P_2 are conjugate points in a rational involution.

The correspondence between the spaces is defined by associating with the planes of (x') a web of surfaces, order n, of (x), such that any three surfaces of the web, not all belonging to the same pencil, intersect in only two variable points. These two points correspond to the one point in (x') which is the intersection of the three associated planes, and they are two conjugate points of the involution. For certain points P' it happens that the images P_1 and P_2 coincide. The locus of such points P' is called the surface of branch points L'; its image, the locus of the point $P_1 \equiv P_2$, is called the surface of coincident points, K. This latter surface is always a component of the Jacobian of the web $| s_n |$, images of the planes of (x').

The first systematic discussion of (1, 2) correspondence between two spaces was given by De Paolis.[*] Previously De Paolis had studied the (1, 2) correspondence between two planes,[†] as had Noether and others.[‡] This space transformation has proved useful in the investigation of the singularities of certain surfaces, for example, the Kummer surface, and has also been used in the study of space involutions by Sharpe and Snyder.[§] In the present paper it is used in discussing some types of monoidal involutions in which the monoidal transformation is compounded with the homology, the Geiser, the Bertini and the Jonquières involutions in the bundle. In addition all monoids of the web defining the (1, 2) correspondence have a fixed tangent cone.

---

[*] R. De Paolis, "Le trasformazioni doppie dello spazio," *Lincei Memorie*, ser. 4, vol. 1 (1885), pp. 576-608.

[†] R. De Paolis, "Le trasformazioni piane doppie," *Lincei Memorie*, ser. 3, vol. 1 (1877), pp. 136-171; "Le trasformazioni piane doppie di seconde ordine e la sua applicazione alla geometria non-euklidea," *Lincei Memorie*, ser. 3, vol. 2 (1878), pp. 31-50; "Le trasformazioni piane doppie di terzo ordine primo genere e la sua applicazione alle curve di quarto ordine," *Lincei Memorie*, ser. 3, vol. 2 (1878), pp. 851-878.

[‡] M. Noether, "Über die ein-zweideutigen Ebenentransformationen," *Sitzungsberichte der physik. med. Societät zu Erlangen*, vol. 10 (1878), pp. 81-86; "Über eine Klasse von auf die einfache Ebene abbildbaren Doppelebenen," *Math. Annalen*, vol. 33 (1889), pp. 525-545.

[§] F. R. Sharpe and V. Snyder, "Certain Types of Involutorial Space Transformations," *Trans. Amer. Math. Soc.*, vol. 20 (1919), pp. 185-202, vol. 21 (1920), pp. 52-78.

II. *The Monoidal Involution Compounded with the Harmonic Homology.*

2. Given a transformation of order n, whose defining equations are

$$(1) \qquad \begin{aligned} x'_i &= p_{n-2}\phi_i \qquad i = 1,\ 2,\ 3. \\ x'_4 &= u_{n-1}x_4 + u_n \end{aligned}$$

where p_{n-2}, u_{n-1}, u_n are ternary in x_1, x_2, x_3, and the net $|\,\phi\,|$ is defined by

$$(1') \qquad \begin{aligned} \phi_1 &= x_1^2 \\ \phi_2 &= x_2x_3 \\ \phi_3 &= x_3^2 \end{aligned}$$

To the planes of (x') correspond a web of n-ics in (x) with the following basis elements: $O^{n-1} \equiv (0001)$, a basis curve $\beta_{n(n-2)}$ which is the intersection $(p_{n-2},\ u_{n-1}x_4 + u_n)$, and the simple point $O_2 \equiv (0100)$ with a consecutive point $O_{2'}$ which lies on the line consecutive to OO_2 in $x_3 = 0$, and also on the monoid $u_{n-1}x_4 + u_n = 0$. All monoids of the web have the common tangent cone $u_{n-1} = 0$.

3. Discussion of the ternary transformation.

$$\begin{aligned} x'_1 &= x_1^2, \\ x'_2 &= x_2x_3, \\ x'_3 &= x_3^2, \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= \sqrt{x'_1x'_3}, \\ x_2 &= x'_2, \\ x_3 &= x'_3. \end{aligned}$$

To any plane $ax'_1 + bx'_2 + cx'_3 = 0$ through $O' \equiv (0001)'$ in (x') corresponds a quadric cone $ax_1^2 + bx_2x_3 + cx_3^2 = 0$ of (x). When the plane describes the bundle (O'), the quadric cones of the corresponding bundle in (x) all pass through the line $OO_2(x_1 = x_3 = 0)$ and have a common tangent plane $x_3 = 0$ along this line.

Consider any pencil in the bundle of planes of O'. There is a pencil of quadric cones in (x) all of which intersect in two lines besides the fixed line OO_2. Therefore to a line of O' correspond two lines of O, and there is a (1, 2) correspondence between two bundles.

The equations of the involution may be written

$$\begin{aligned} x'_1 &= -x_1, & & & x'^2_1 &= x_1^2, \\ x'_2 &= x_2, & &\text{by solving} & x'_2x'_3 &= x_2x_3, \\ x'_3 &= x_3, & & & x'^2_3 &= x_3^2. \end{aligned}$$

The fundamental elements in the (1, 2) correspondence are

$$x_1 = x_3 = 0, \qquad x'_2 = x'_3 = 0, \qquad x_3 = 0.$$

The line $x_2 = x_3 = 0$ corresponds to $x'_2 = x'_3 = 0$, a basis line in (x'). As a point approaches $x'_2 = x'_3 = 0$ in any plane $x'_2 + \lambda x'_3 = 0$, its two images in (x) consist of a direction in the plane $x_2 + \lambda x_3 = 0$, and the plane $x_3 = 0$. Hence the true image of $x'_2 = x'_3 = 0$ is the line $x_2 = x_3 = 0$; x_3 is an extraneous factor.

Similarly a direction thro $x_1 = x_3 = 0$ has for image a direction thro $x'_1 = x'_3 = 0$ and the extraneous factor x'_3. The proper image in (x') of $x_1 = x_3 = 0$ is the line $x'_1 = x'_3 = 0$. To a direction through $x'_1 = x'_3 = 0$ correspond two directions through $x_1 = x_3 = 0$. Hence the image of every line through O' in $x'_3 = 0$, excluding $x'_2 = x'_3 = 0$, is the line $x_1 = x_3 = 0$.

The tangent plane $x_3 = 0$ has for image in (x') the line $(x'_2 = x'_3 = 0$, which is basis for the cones, images in (x') of the planes of O.

The equation of L' is $x'_1 = 0$.

From a consideration of the equations of the involution, it is clear that K is the plane $x_1 = 0$.

4. Now return to the general transformation given by equations (1). A line through O pierces a monoid of the web in a single point P_1 apart from O, to which corresponds in (x') a point P', the intersection of the line associated with OP_1, with the plane which is the image of the monoid. The line $O'P'$ corresponds to two lines in (x), OP_1 and a second line intersecting the monoid at the point P_2. Hence there is a (1, 2) correspondence between the points of two spaces; P_1 and P_2 are conjugate points in the involution I.

Two surfaces of the web intersect in a curve of order n^2, composed of $\beta_{n(n-2)}: O^{(n-1)(n-2)}$ and a variable curve c_{2n}. Hence the following correspondence exists:

$$s_1 \sim s'_{2n}; \quad c_1 \sim c'_n; \quad s'_1 \sim s_n; \quad c'_1 \sim c_{2n}.$$

c_{2n} has O to multiplicity $2(n-1)$. The projecting cone of c_{2n} from O is a quadric belonging to the $|\phi|$ net, and each generator meets c_{2n} in only one point apart from O. Hence genus of c_{2n} equals genus of a quadric cone;

$p = 0$. This result is not in keeping with the Riemann-Roch theorem, since $p_a = 0$.

5. The order of L' is given by the value of $2p + 2$, but when $p = 0$, this formula cannot be applied. However, it is clear geometrically that L' is identical with the surface of branch points for the ternary transformation, that is, the plane $x'_1 = 0$. For to any point P' on L' correspond two coincident points P on K. Hence the line $O'P'$ corresponds to a double line OP, and to every point on $O'P'$ corresponds a repeated point on OP. The entire line $O'P'$ lies on L', and the locus of points which have coincident images in (x) is evidently identical with the locus of lines whose images in the ternary transformation are coincident lines. The surface of coincidences K is the plane $x_1 = 0$, and is a component of the Jacobian $(J = 4x_1x_3^2u_{n-1}p_{n-2}^2)$ of the web (1). In addition the line $x_2 = x_3 = 0$ is invariant in I; this is evident from the equations of the involution (2) since either x_1 and x_4 both change sign, or both remain the same, according as n is odd or even.

6. Images in (x') of fundamental elements of (x). All points on the line OO_2 except O and O_2 have for image the point O'. If a point of OO_2 is approached in the plane $x_1 + \lambda x_3 = 0$, the image point approaches O' on the locus $x'_3(x'_1 - \lambda^2 x'_3) = 0$. Hence the points of this line are in $(1, 1)$ correspondence with the pencil of lines through O' in $x'_3 = 0$. The image of O_2 is the line $x'_1 = x'_3 = 0$.

7. The image of every point on a simple basis curve is a straight line, so that the image in (x') of $\beta_{n(n-2)}$ is a ruled surface. Any c_{2n} meets p_{n-2} in $2n(n-2)$ points which must lie on β. Of these, $2(n-1)(n-2)$ are at O; hence c_{2n} meets β in $2(n-2)$ variable points, corresponding to $2(n-2)$ points in which a straight line of (x') meets the image of β. All points of p_{n-2} except basis points have O' for image; any curve on p_{n-2} corresponds to a bundle of directions through O'. For the basis curve, these directions are given by the lines of the ruled surface, which is therefore a cone $R'_{2(n-2)}$.

A plane through O cuts c_{2n} in two variable points. Therefore the surface which is the image in (x') of this plane meets a straight line in two variable points and is a quadric surface Q'_2. The complete image must be of order $2n$, so that the image in (x') of O is a fixed surface $Q'_{2(n-1)}$. Since the equation of a plane through O does not involve x_4, the image does not involve x'_4, and both Q'_2 and $Q'_{2(n-1)}$ are cones.

8. Every generator of the common tangent cone $u_{n-1} = 0$ is a parasitic line. If a surface of the web contains an additional point on a generator,

it contains the entire generator, and there is a net of surfaces of the web containing the same line. This net corresponds to a bundle of planes in (x'); the generator corresponds to the vertex of the bundle. As the generator describes the cone u_{n-1}, the image point describes a curve ξ'. The cone u_{n-1}, cuts a surface of the web in $n(n-1)$ lines; this is the order of ξ', since it is the number of points in which a plane of (x') cuts ξ'. Any s_1 not through O cuts u_{n-1} in all its generators, so that every s'_{2n} image of a plane of (x), passes through ξ_1. For the composite s'_{2n}, ξ' lies on the fixed component $Q'_{2(n-1)}$. The $(n-1)(n-2)$ intersections (u_{n-1}, p_{n-2}) correspond to $(n-1)(n-2)$ directions through O', so that $\xi'_{n(n-1)}$ has $(O')^{(n-1)(n-2)}$. Hence each generator of $Q'_{2(n-1)}$ meets ξ' once apart from O', and the genus of ξ' is equal to that of the cone. A study of the involution will show that the image of ξ' in (x) is composite, being u_{n-1} and a basis curve γ; the genus of ξ' is therefore equal to that of $u_{n-1}=0$ which, with no restrictions, is $\frac{(n-2)(n-3)}{2}$. This shows that $Q'_{2(n-1)}$ has $\frac{3(n-1)(n-2)}{2}$ double generators. Of these, $\frac{(n-1)(n-2)}{2}$ result from pairs of conjugate lines on u_{n-1}. The other $(n-1)(n-2)$ coincide along $x'_2=x'_3=0$, for performing the ternary transformation on $u_{n-1}=0$, it is found that $Q'_{2(n-1)}$ has the line $x'_2=x'_3=0$ to multiplicity $n-1$, with the plane $x'_3=0$ tangent to all $n-1$ branches.

9. Equations of the involution. The ternary involution gives $x_1=-x'_1$, $x_2=x'_2$, $x_3=x'_3$, and also $\phi_i(x)=\phi_i(x')$ for all ϕ of the net. Denote $f(-x'_1x'_2x'_3)$ by $\bar{f}$ and $f(x'_1x'_2x'_3)$ by f'. Then the equations of the involution are given by

$$\frac{u_{n-1}x_4+u_n}{p_{n-2}\phi_i}=\frac{u'_{n-1}x'_4+u'_n}{p'_{n-2}\phi_i}$$

which becomes

$$x_4=\frac{\bar{p}_{n-2}u'_{n-1}x'_4+\bar{p}_{n-2}u'_n-p'_{n-2}\bar{u}_n}{p'_{n-2}\bar{u}_{n-1}}$$

Interchanging primed and unprimed letters, the equations of the involution become

$$(2)\qquad \begin{aligned} x'_1&=-p_{n-2}\bar{u}_{n-1}x_1,\\ x'_2&=p_{n-2}\bar{u}_{n-1}x_2,\\ x'_3&=p_{n-2}\bar{u}_{n-1}x_3,\\ x'_4&=\bar{p}_{n-2}(u_{n-1}x_4+u_n)-p_{n-2}\bar{u}_n. \end{aligned}$$

The involution is of order $2(n-1)$; the Jacobian of the system is $J_I=$

$\bar{p}_{n-2}\bar{u}^3_{n-1}p^3_{n-2}u_{n-1}$; every surface is a cone or a monoid, and all monoids have the common composite tangent cone $\bar{p}_{n-2}u_{n-1}=0$.

The basis elements of the involution are as follows:

Lines $(p_{n-2},\ \bar{p}_{n-2})$; of these $n-2$ are invariant on K, and $(n-2)(n-3)$ are arranged in pairs of conjugates	$(n-2)^2$
Curve $\beta_{n(n-2)}$: $(p_{n-2},\ u_{n-1}x_4+u_n)$	$n(n-2)$.
Lines $(K,\ u_{n-1})$	$n-1$
Curve $\gamma_{(n-1)(2n-3)}$; given by the intersection $(\bar{u}_{n-1},\ \bar{p}_{n-2}[u_{n-1}x_4+u_n]-p_{n-2}\bar{u}_n)$ after subtraction of the lines $(K_1,\ u_{n-1})$	$(n-1)(2n-3)$.

The total fixed intersection of two monoids of the web (2) is of order $[2(n-1)]^2-2(n-1)$; the variable curve of intersection is of order $2(n-1)$.

Note that the point O_2 and the line OO_2 are not basis in I. O is basis to multiplicity $2n-3$.

10. Discussion of the involution. The conjugate of a plane $ax_1+bx_2+cx_3=0$ is a plane $ax_1-bx_2-cx_3=0$. These meet in one invariant line which is also on $K_1: x_1=0$. An arbitrary plane through O does not contain any pairs of conjugate lines, but all planes of the form $x_2+mx_3=0$ are invariant in I; the pencil of lines through O in any one of them is self-conjugate, a line of the pencil having for its conjugate another line of the pencil.

By the involution, $u_{n-1}=0$ is transformed into $\bar{u}_{n-1}=0$. Every monoid meets a generator r of u_{n-1} only at O. It meets the conjugate line r' at some point not O which varies with the monoid, so that to the point O on r corresponds the entire line r', and the conjugate of O alone is $\bar{u}_{n-1}=0$. From equations (2), it is evident that the complete conjugate of O is $p_{n-2}\bar{u}_{n-1}=0$.

The intersection $(u_{n-1},\ \bar{u}_{n-1})$ consists of $n-1$ lines $(K_1,\ u_{n-1})$ and $\dfrac{(n-1)(n-2)}{2}$ pairs of conjugates. Since every line of $u_{n-1}=0$ is parasitic, the conjugate of $u_{n-1}=0$ is a curve on $\bar{u}_{n-1}=0$, which, since any plane meets u_{n-1} in all its generators, must lie on all surfaces of the web (2). The cone $u_{n-1}=0$ meets $s_{2(n-1)}$ in $(n-1)(2n-3)$ lines apart from $(K_1,\ u_{n-1})$, the images of which are $(n-1)(2n-3)$ points in a plane. The basis curve is therefore of order $(n-1)(2n-3)$; call it $\gamma_{(n-1)(2n-3)}$.

The points of γ which correspond to the $(n-1)(n-2)$ conjugate lines $(u_{n-1}, \bar{u}_{n-1})$ are all at O, since to a generator of $\bar{u}_{n-1}=0$ corresponds a direction through O; $(n-1)(n-2)$ lines (p_{n-2}, u_{n-1}) also correspond to directions through O; hence γ has O to multiplicity $2(n-1)(n-2)$.

The lines (K_1, u_{n-1}) are at the same time invariant and parasitic. They lie on $\bar{u}_{n-1}$, so that each of them meets γ in one point not at O. An arbitrary plane π meets any one of these lines in a point P. Since this joint is on K, the monoid of the web $|s_{2(n-1)}|$ corresponding to π passes through P. It also passes through γ which is basis for the web; hence this generator has $2n-3$ intersections with the monoid at O, 1 on γ, and 1 at P, a total of $2n-1$. The generator lies entirely on this arbitrary monoid, and is therefore basis for the web. A similar proof holds for all the lines (K_1, u_{n-1}).

The variable curve $c_{2(n-1)}$, conjugate of an arbitrary c_1 in I, has O to multiplicity $2n-3$, for since a straight line meets a plane through O in one point, the corresponding plane through O meets $c_{2(n-1)}$ in one variable point. $c_{2(n-1)}$ meets $\beta_{n(n-2)}$ in $n-2$ points apart from O, for it meets p_{n-2} in this number apart from O, and since $c_{2(n-1)}$ is the intersection of two $s_{2(n-1)}$ which have their entire intersection with p_{n-2} fixed, the $n-2$ points must lie on basis elements. An arbitrary line does not meet $\bar{p}_{n-2}$ in any of the $(n-2)^2$ basis lines; therefore the points of intersection of $c_{2(n-1)}$ and p_{n-2} lie on $\beta_{n(n-2)}$. Since a line meets u_{n-1} in $n-1$ points, $c_{2(n-1)}$ meets $\gamma_{(n-1)(2n-3)}$ in $n-1$ points, and finally, since a line meets a plane on one point, $c_{2(n-1)}$ meets any monoid of the web $|s_{2(n-1)}|$ in one variable point.

11. Double curve. The intersection $(s_1, s_{2(n-1)})$ is composed of the line (s_1, K_1) and a residual δ_{2n-3}, the image of which is the double curve on s'_{2n}. This double curve is of order $\dfrac{n(n-1)}{2}$ and genus $(n-2)(n-3)$.

The fundamental elements u_{n-1}, $x_3{}^{2n}$, $p_{n-2}{}^{2n-1}$ are of orders $n-1$, $2n$ and $2n^2-5n+2$ respectively.

This case, where the monoidal involution is compounded with the harmonic homology, was thought by Martinetti to be the only possible non-perspective monoidal involution.*

III. *The Monoidal Involution Compounded with the Geiser.*

12. The defining equations of this case are

---

* V. Martinetti, "Sopra una classe di trasformazioni involutorie dello spazio," *Istituto Lombardo Rend.*, ser. 2, vol. 18 (1885), pp. 132-142.

(3) $$x'_i = p_{n-3}\phi_i \qquad i = 1,\ 2,\ 3.$$ $$x'_4 = u_{n-1}x_4 + u_n$$

where p_{n-3}, u_{n-1}, and u_n are all ternary in x_1, x_2, x_3, and the $\phi_i = 0$ are cubic cones with 7 common generators $OP_1, \cdots, OP_7$, and vertex at (0001).

The web of n-ics corresponding to the planes in (x') space has as basis elements; O^{n-1}, 7 points P_i lying simply on each ϕ_i and on $u_{n-1}x_4 + u_n = 0$, and the curve $\beta_{n(n-3)}$ which is the intersection $(p_{n-3},\ u_{n-1}x_4 + u_n)$.

13. Discussion of the ternary transformation.

$$x'_i = \phi_i \qquad i = 1,\ 2,\ 3.$$

The basis elements are the 7 lines OP_i; call them h_i. This is recognized as the Geiser transformation in the bundle. The images of the 7 basis lines in (x) are 7 planes in (x').

A plane π in (x) through O cuts any cubic cone of the net in three generators; hence a plane in (x') through O' cuts the image of π in three lines. This image is therefore a rational cubic cone of (x').

L' is a cone of order 4. The image of L'_4 is the surface of coincidences, counted twice; it is a sextic cone K_6 with the $7h_i$ as double lines, and is the Jacobian of the net of ϕ's.

Each of the 7 planes in (x'), images of the 7 basis lines, has for image in (x) the line h_i and a nodal cubic cone $\psi_i : h_i^2$. Therefore the 7 planes are tangent to L' wherever they meet it, the necessary and sufficient condition for tangency to L' being that the image of the locus shall be composite in (x) space. Since L' is of order 4, the planes are bitangents to the cone. Each of the 7 planes cuts every other one in a line; the cone ψ_i passes once through each of the 6 basis lines h_k $(k \neq i)$ and twice through h_i because of the bitangency.

s_1 corresponds to a cone G_8 in I. G_8 has the $7h_i$ as triple lines. The genus of G_8 is zero, the same as its conjugate plane in the involution.

The Jacobian of the net $| G_8 |$ is composed of the 7 nodal cubic cones ψ_i. It is of order 21, and contains the basis lines 8-fold. It will be referred to as J_{21}.*

14. Returning to the general transformation (3), we see that the planes of (x') correspond to the surfaces of a web of monoids, vertex at

---

* V. Snyder, "Conjugate Line Congruences Contained in a Bundle of Quadric Surfaces," *Trans. Amer. Math. Soc.*, vol. 11 (1910), pp. 371-387.

(0 0 0 1), with a fixed tangent cone. A line through O' corresponds to two lines through O. The two lines through O pierce a monoid of the web in two points corresponding to the one point in which the line of O' meets the plane which is the image in (x') of the monoid.

Two monoids of the web intersect in a basis curve $\beta_{n(n-3)} : O^{(n-1)(n-3)}$, and a variable curve $c_{3n} : O^{3(n-1)}$. The following correspondences exist:

$$s_1 \sim s'_{3n}; \quad c_1 \sim c'_n; \quad s'_1 \sim s_n; \quad c'_1 \sim c_{3n}.$$

Since c_{3n} has $O^{3(n-1)}$, its projecting cone is a cubic, no generator of which meets c_{3n} in more than one point. This cubic cone belongs to the net $|\phi|$. The curve has no apparent double points and hence the genus of c_{3n} is the genus of a non-singular cubic cone; $p = 1$. This agrees with the Riemann-Roch theorem for a regular system.

The order of L' is therefore 4. It is the cone L'_4 associated with the Geiser transformation in the bundle, and the surface of coincidence is the corresponding cone K_6. This can be proved by an argument identical with that given in Section II.

15. Images in (x') of the fundamental elements of the web. The image of each point P_i is a bitangent plane to L'_4. To show that the image of $P_i \equiv (y)$ is a plane, take the point $(y_i + \epsilon\xi_i)$ and expand the expressions for x'_i in powers of ϵ. We obtain

$$x'_1 = \epsilon \left(\frac{\partial p_{n-3}\phi_1}{\partial x_1} \xi_1 + \frac{\partial p_{n-3}\phi_1}{\partial x_2} \xi_2 + \frac{\partial p_{n-3}\phi_1}{\partial x_3} \xi_3 \right) y_i + \cdots$$

and similar expressions for the other x'_i. Divide by ϵ and then let ϵ approach zero. Eliminating ξ_1, ξ_2, ξ_3, between the four expressions, we get a linear equation in x'_1, giving a plane as the image of P_i.

Any point on h_i except P_i and O has for image in (x') the point O'; corresponding to each of these points on h_i, there is a pencil of directions through O', one direction corresponding to each plane of approach to the line. For a plane π through h_i has as image in (x') a bitangent plane of L'_4 corresponding to P_i, and a quadric cone. These intersect in two lines, one corresponding to the line consecutive to h_i in π, the other to that intersection of π with ψ_i which is not at h_i, where ψ_i is the nodal cubic cone already referred to. As a point approaches h_i along a line of π, the image point approaches O' on a line common to the bitangent plane and the quadric cone. This direction through O' varies with the quadric, and hence with the plane π but always lies in the bitangent plane which is the image of P_i.

The image in (x') of $\beta_{n(n-3)}$ is a ruled surface. The curve c_{3n} meets $\beta_{n(n-3)}$ in $3(n-3)$ points apart from O', so that a straight line in (x') meets the image of $\beta_{n(n-3)}$ in $3(n-3)$ points. Any point of p_{n-3} except basis points has O' for image, so that a curve on p_{n-3} corresponds to a cone of directions through O'; hence the ruled surface is a cone $R'_{3(n-3)}$. It will appear later that this surface is a component of the tangent cone to s'_{3n} at O'.

A plane through O meets c_{3n} in 3 variable points. Hence the s'_{3n} which is the image of this plane meets a straight line of (x') in three variable points, so that the image is composite. It consists of a fixed cone $Q'_{3(n-1)}$, image of O, and a variable cone Q'_3.

By reasoning identical with that given in Section II, the image of the tangent cone $u_{n-1}=0$ is known to be a curve $\xi'_{n(n-1)}$, basis for all s'_{3n} and lying on the fixed cone $Q'_{3(n-1)}$. The intersections of $u_{n-1}=0$, $p_{n-3}=0$, correspond to $(n-1)(n-3)$ directions through O'; ξ' has O' to multiplicity $(n-1)(n-3)$. Hence each generator of $Q'_{3(n-1)}$ meets ξ' once apart from O', and the genus of ξ' is equal to that of the cone. As in the previous case, this is equal to $\frac{(n-2)(n-3)}{2}$. The $(n-1)(4n-7)$ double lines on $Q'_{3(n-1)}$ result from that number of pairs of conjugate lines on u_{n-1}.

16. Equations of the involution. If we use the notation ϕ'_i, etc., to indicate the conjugate of ϕ_i in (x), we have

$$\text{(4)} \qquad \begin{aligned} p_{n-3}\phi_i &= p'_{n-3}\phi'_i \qquad i=1,\ 2,\ 3.\\ u_{n-1}x_4+u_n &= u'_{n-1}x'_4+u'_n. \end{aligned}$$

From the result of the Geiser transformation, we have

$$x'_i = G_i \qquad i=1,\ 2,\ 3.$$

where G_i is of order 8, and also

$$\phi'_i = \phi_i(G_i) = \phi_i J_{21}.$$

Making these substitutions, and solving equations (4) for x'_i, we find for the equations of the involution

$$\text{(5)} \qquad \begin{aligned} x'_i &= p_{n-3}u_{n-1}(G_i)G_i \qquad i=1,\ 2,\ 3.\\ x'_4 &= p_{n-3}(G_i)J_{21}(u_{n-1}x_4+u_n)-p_{n-3}u_n(G_i). \end{aligned}$$

It is of order $3(3n-1)$; the lines h_i are basis of order $3n-1$; the points P_i are basis of order $3n$. The tangent cone of the web of $s_{3(3n-1)}$ is fixed and composite, $p_{n-3}(G_i)J_{21}u_{n-1}=0$. Other basis elements are:

The curve $\beta_{n(n-3)} \equiv (p_{n-3},\ u_{n-1}x_4+u_n)$.

The lines $(p_{n-3},\ p_{n-3}(G_i))$. Since these surfaces are conjugate in I, their lines of intersection must be either invariant, or conjugate lines. The intersection $(K_6,\ p_{n-3})$ gives $6(n-3)$ invariant lines; the remaining $2(n-3)(4n-15)$ are arranged in pairs of conjugates.

The lines $(p_{n-3},\ J_{21})$.

The lines $(K_6,\ u_{n-1})$, which are also on $u_{n-1}(G_i)$.

The curve $(u_{n-1}(G_i),\ p_{n-3}(G_i)J_{21}\ (u_{n-1}x_4+u_n)-p_{n-3}u_n(G_i))$.

From this intersection must be subtracted the lines h_i, and the lines $(K_6,\ u_{n-1})$, both of which have been counted elsewhere. The residual intersection is of order $9(n-1)^2$. Call it $\gamma_{9(n-1)^2}$.

The total fixed intersection of two surfaces $s_{3(3n-1)}$ is as follows:

The lines h_i count for an intersection of order........	$7(3n-1)^2$
Lines consecutive to h_i, due to tangency.............	$7(3n-1)$
$\beta_{n(n-3)}$	$n(n-3)$
$(p_{n-3},\ p_{n-3}(G_i))$..................................	$8(n-3)^2$
$(K_6,\ u_{n-1})$...	$6(n-1)$
$(p_{n-3},\ J_{21})$..	$21(n-3)$
$\gamma_{9(n-1)^2}$	$9(n-1)^2$

The variable residual intersection is a curve of order $3(3n-1)$; it is the conjugate of a straight line.

$$J_I = u_{n-1}p^3_{n-3}\ [u_{n-1}(G_i)]^3\ p_{n-3}(G_i)J_{21}^2.$$

17. Discussion of the involution. The conjugate of any plane s_1 through O is a cone G_8, meeting the plane in 8 lines. Six of them are the invariant intersections $(s_1,\ K_6)$; the remaining two are conjugate to each other. Hence every plane through O contains one and only one pair of conjugate lines.

The cone $u_{n-1}(G_i)=0$, obtained by performing the involution on $u_{n-1}=0$, is conjugate to O in I. The complete conjugate of O is seen from equations (5) to be $p_{n-3}u_{n-1}(G_i)=0$. The conjugate of u_{n-1} is a curve γ, which is basis in I and lies on $u_{n-1}(G_i)$. Since the cone u_{n-1} meets $s_{3(3n-1)}$

in $9(n-1)^2$ lines, apart from the fundamental lines (K_6, u_{n-1}), the order of γ is $9(n-1)^2$. It is the curve

$$[u_{n-1}(G_i),\ p_{n-3}(G_i)J_{21}\ (u_{n-1}x_4 + u_n) - p_{n-3}u_n(G_i)].$$

The points of γ which are images of the $2(n-1)(4n-7)$ conjugate lines $(u_{n-1}, u_{n-1}(G_i))$ are all at O; the intersections (p_{n-3}, u_{n-1}) also correspond to directions through O; hence γ has O to multiplicity $(n-1)(9n-17)$. The $3(n-1)$ generators common to u_{n-1} and ψ_i have for images $3(n-1)$ directions through P_i; hence γ has the 7 points P_i to multiplicity $3(n-1)$.

The argument of Section II shows that the lines (K_6, u_{n-1}) are basis in the involution.

18. Let A be an arbitrary point on h_i and let a line c_1 meet h_i at A. The point P_i has for image in (x') a plane bitangent to L'_4. From the Geiser transformation it is known that the image in (x) of this bitangent plane is a cubic cone ψ_i with h_i double and the $6h_k$ $(k \neq i)$, simple. Hence the conjugate of P_i in the involution is ψ_i. Since the variable conjugate of any plane through O is a G_8 with h_i^3 $(i = 1, 2, \cdots, 7)$, the conjugate of the plane through c_1 and h_i is a quintic cone Q_5 passing simply through h_i and doubly through the $6h_k$. A line through O in this plane meets c_1 once, so that a generator of Q_5 meets $\bar{c}$, the conjugate of c_1 once. The image of any point on ψ_i, avoiding the basis lines, is P_i; ψ_i and Q_5 have 14 lines of intersection along the basis lines; the fifteenth is a line $\bar{g}$ conjugate to g, the line through O consecutive to h_i in the plane (c_1, h_i). When the point generating c_1 passes through A, its conjugate describing $\bar{c}$ will pass through $\bar{A}$ on $\bar{g}$. Since every point on $\bar{g}$ except O is conjugate to P_i, $\bar{A}$ must be at O, that is, one branch of $\bar{c}$ at O has $\bar{g}$ for tangent. The line c_1 meets $p_{n-3}u_{n-1}(G_i)$ in $3(n-1)$ points at A, and in $6n-8$ other points. The conjugate of each of these points is at O, hence $\bar{c}$ has O to multiplicity $6n-7$. Since it lies on Q_5, its order is $2(3n-1)$. It has been seen that the order of h_i in the involution is $3n-1$, so that this result checks with the fact that the conjugate of an arbitrary line is of order $3(3n-1)$.

This discussion shows that the conjugate of any point of h_i is the entire line h_i to multiplicity $3n-1$. These lines, isolated parasitic lines for the (1, 2) correspondence, are fundamental of the second kind in the involution.* It is also clear that each point on h_i corresponds to ∞^1 directions through O on the cone ψ_i. There is a (1, 1) correspondence between these directions and the pencil of planes through h_i.

---

* D. Montesano: "Sulla teoria generale delle corrispondenze birazionali dello spazio," Nota III, *Lincei Rend.*, ser. 5, vol. 30 (1921), pp. 447-451.

19. The complete conjugate of a plane through O is a cone of order $3(3n-1)$, having each h_i to multiplicity $3n$. The monoids of (5) contain these lines to multiplicity $3n-1$. Since the intersection of two monoids can always be reduced to the intersection of a cone and a monoid, h_i must be of multiplicity $3n(3n-1)$ in the intersection, so that along each h_i there are $3n-1$ consecutive lines common to all monoids of the web. The $3n-1$ planes tangent to all monoids along h_i are those of the fixed tangent cone of the web, $p_{n-3}(G_i)J_{21}u_{n-1}=0$. The cone u_{n-1} does not contain h_i, but $p_{n-3}(G_i)$ has $h_i^{3(n-3)}$, thus determining $3(n-3)$ fixed tangent planes along h_i, and the other 8 are given by J_{21}, 2 by ψ_i, and 1 by each of the $6\psi_k$. Hence the fixed tangent cone of the web (5) is not only tangent to all surfaces of the web at O, but is also tangent to the $3n-1$ branches of every surface along each of the 7 lines h_i.

The variable curve $c_{3(3n-1)}$, conjugate of an arbitrary c_1, has O to multiplicity $9n-11$, for since a straight line meets a G_i in 8 points, the corresponding $c_{3(3n-1)}$ meets a plane through O in 8 variable points. All tangents to the curve at O lie on the same fixed cone. The curve passes through each of the $7P_i$ to multiplicity 3, since a line meets ψ_i in 3 points; it meets $\gamma_{9(n-1)^2}$ in $n-1$ points, and meets $\beta_{n(n-3)}$ in $8(n-3)$ points apart from O. For it meets p_{n-3} in $8(n-3)$ points and since it is the intersection of two $s_{3(3n-1)}$ which have their entire intersection with p_{n-3} fixed, the $8(n-3)$ points must lie on basis elements. An arbitrary line will not meet $p_{n-3}(G_i)$ in any of the $8(n-3)^2$ lines $(p_{n-3}, p_{n-3}(G_i))$, neither will it pass through any of the 7 basis points whose images are the 7 nodal cubics composing J_{21}; hence the points of intersection of $c_{3(3n-1)}$ and p_{n-3} avoid the lines $(p_{n-3}, p_{n-3}(G_i))$ and (p_{n-3}, J_{21}), and must lie on $\beta_{n(n-3)}$. Finally, the curve $c_{3(3n-1)}$ meets any monoid of the web $|s_{3(3n-1)}|$ in one variable point.

20. Double Curve. The intersection $(s_1, s_{3(3n-1)})$ is composed of (s_1, K_6) and a residual $\delta_{9(n-1)}$. Since $s_{3(3n-1)}$ has h_i^{3n-1}, and K_6 has h_i^2, $\delta_{9(n-1)}$ has 7 points $H_i^{3(n-1)}$. The plane s_1 intersects $\beta_{n(n-3)}$ in $n(n-3)$ points which are on $s_{3(3n-1)}$, and therefore on $\delta_{9(n-1)}$. The curves (s_1, K_6) and $\delta_{9(n-1)}$ intersect in $12(n-1)$ points apart from $7H_i$. These points have images on L'_4 which are branch points of the image of $\delta_{9(n-1)}$ in (x'), a double curve on s'_{3n}.

Cut $\delta_{9(n-1)}$ by an arbitrary s_n of the web (3). The plane of δ meets s_n in a curve c_n through $n(n-3)$ points of $\beta_{n(n-3)}$ and therefore δ meets s_n in $2n(4n-3)$ other points, arranged in pairs of conjugates, which are images of $n(4n-3)$ points in which a plane of (x') meets δ', the double curve. The order of δ' is therefore $n(4n-3)$.

The genus of $\delta_{9(n-1)}$ is $9n^2-21n+13$. To find the genus of $\delta'_{n(4n-3)}$, apply Zeuthen's formula $\eta-\eta'=2a(p'-1)-2\beta(p-1)$; $\eta=0$, $\eta'=12(n-1)$, $a=2$, $\beta=1$, $p=9n^2-21n+13$. Solving for p', we find for the genus of δ' the value $\dfrac{(3n-4)(3n-5)}{2}$.

21. Fundamental elements extraneous in the involution. Consider a plane $\sum\limits_1^3 a_i x_i = 0$ through O. Its image in (x') is a fixed $Q'_{3(n-1)}$ and a variable Q'_3; its conjugate in I is $p_{n-3}u_{n-1}(G_i)\ \sum\limits_1^3 a_i G_i = 0$. If the values (3) for x'_1, x'_2, x'_3, are substituted in $Q'_{3(n-1)}\ Q'_3$, p^{3n}_{n-3} is a factor of the resulting equation. Hence p^{3n-1}_{n-3} is extraneous in I. In addition the transformed $Q'_{3(n-1)}$, which is now of order $9n-11$, has lost a factor of order $n-1$. This factor represents the cone u_{n-1}, the common tangent cone of the web $|\,s_n\,|$. This is evident when it is recalled that every s'_{3n}, and for composite s'_{3n} the fixed $Q'_{3(n-1)}$, passes through $\xi'_{n(n-1)}$, the image of u_{n-1} in (x').

Since these components are extraneous for an arbitrary s'_{3n}, they are extraneous for every s'_{3n}. This fact may be used in investigating the general s'_{3n}.

Given a plane $\sum\limits_1^4 a_i x_i = 0$ in (x). Its image in (x') is a certain s'_{3n}; its conjugate in the involution is

$$p_{n-3}u_{n-1}(G_i)\ \sum_1^3 a_i G_i + a_4\ \left[p_{n-3}(G_i)J_{21}(u_{n-1}x_4+u_n) - p_{n-3}u_n(G_i)\right] = 0$$

$$\therefore\ s'_{3n}(p_{n-3}\phi_1,\ p_{n-3}\phi_2,\ p_{n-3}\phi_3,\ u_{n-1}x_4+u_n) =$$

$$p_{n-3}u_{n-1}\ \sum_1^4 a_i x_i\ [p_{n-3}u_{n-1}(G_i)\ \sum_1^3 a_i G_i + a_4\ \{p_{n-3}(G_i)J_{21}\cdot$$

$$u_{n-1}x_4+u_n) - p_{n-3}u_n(G_i)\}].$$

The highest power of x_4 on the right hand side of this equation is $x_4{}^2$. Since x_4 enters only in the expression for x'_4, and there linearly s'_{3n} must have x'^2_4 as highest power of x'_4, so that O' is a point of multiplicity $3n-2$. Hence we may write:

$$s'_{3n} = u'_{3n-2}x'^2_4 + v'_{3n-1}x'_4 + w'_{3n} = 0. \tag{6}$$

The same result can be obtained geometrically. Any line r' of the bundle O' has two image lines r_1 and r_2 in (x). They intersect a plane s_1 in two points A_1, A_2, which are not conjugate, and hence have two distinct image points on r', A'_1, and A'_2. The two images of A'_1 are conjugates in I; one is on r_1 and s_1, the other on r_2 and $s_{3(3n-1)}$. The same is true for A'_2. Since every line of the bundle O meets s_1 in only one point, r' can have but one pair of points on s'_{3n}, and therefore s'_{3n}, has O'^{3n-2}.

To determine the tangent cone of s'_{3n}, equate the coefficients of $x^2{}_4$ in the two expressions for s'_{3n}.

$$u'_{3n-2}(p_{n-3}\phi_1,\ p_{n-3}\phi_2,\ p_{n-3}\phi_3)\ u^2{}_{n-1} = a^2{}_4 p\,{}^{3n-1}_{n-3}\, u^2{}_{n-1} p_{n-3}(G_i) J_{21}$$
$$\therefore\ u'_{3n-2}(\phi) = a^2{}_4 p_{n-3} p_{n-3}(G_1) J_{2i}.$$

The tangent cone of s'_{3n} is therefore fixed. Its components are the 7 bitangent planes corresponding to J_{21}, and the cone $R'_{3(n-3)}$, the image in (x') of $\beta_{n(n-3)}$, since the image in (x) of $R'_{3(n-3)}$ is $p_{n-3}p_{n-3}(G_i)$.

This case for the particular value $n = 3$, has already been treated.*

IV. *The Monoïdal Involution Compounded with the Bertini.*

22. The transformation in this case is based on the ternary involutorial transformation given by a net of sextic cones with 8 double lines $OP_1, \cdots, OP_8$, and 2 simple lines OP' and OP''.

The ternary type was first studied by Bertini.† Through 8 arbitrary lines $OP_1, \cdots, OP_8$ there are ∞^3 sextic cones which have these for double lines. If the condition is imposed that cones of the system pass through a line OP', it is found that every cone of the net thus obtained passes through another fixed line OP''. The expression for this line has been derived.‡ The (1, 2) correspondence between the lines of (O') in (x') and (O) in (x) is given by the equations

$$x'_1 = \phi^2, \quad x'_2 = \phi\psi, \quad x'_3 = f, \tag{7}$$

---

* C. Moffa, "Su alcune corrispondenze birazionali involutorie dello spazio dotate di un sistema lineare di dimensione tre di superficie del terzo ordine unite," Naples, 1923, Unione Tipografica Combattenti.

† E. Bertini: "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Mat.*, ser. 2, Vol. 8 (1877), pp. 244-286.

‡ V. Snyder, "The Involutorial Birational Transformation of the Plane, of Order 17," *Amer. Journ. Math.*, vol. 33 (1911), pp. 327-336.

in which $\phi = 0$, $\psi = 0$ are cubic cones through $OP_1, \cdots, OP_8$ and $f = 0$ is a proper sextic cone having these lines double.

These three sextic cones have as basis lines $OP^2_1, \cdots, OP^2_8$, and in addition the other two lines common to f and ϕ, which will be called OP' and OP''. In other words, we consider a net contained in the triply infinite system of Bertini sextics.

The locus of coincidence points is a cone of order 9, whose equation is

$$\frac{\partial(\phi\psi f)}{\partial(x_1 x_2 x_3)} = 0. \tag{8}$$

This K_9 has $OP^3_1, \cdots, OP^3_8$. The order of the surface of branch points L' is 6.* It has three branches with a common tangent plane.

The image in (x) of any plane through O' is

$$a_1\phi^2 + a_2\phi\psi + a_3 f = 0.$$

The image in (x') of a plane through O is also a sextic cone, which is rational, since its generators have a (1, 1) correspondence with the pencil of lines in the plane through O. Noether showed that these sextics of (x') form an ∞^2 quadratic system, and that they have three branches through the line $x'_1 = x'_2 = 0$ with the common tangent plane $x'_1 = 0$.

Since to a line of (x') through O' corresponds a pair of lines of O, an involution is established between the lines of O. The conjugate of a plane in this involution is known to be a cone of order 17, having each line OP_i six-fold. Call this cone B_{17}.

The conjugates of the lines OP_i in I are sextic cones σ_i, such that σ_i has $OP_i{}^3$ and $OP_k{}^2$ $(k \neq i)$. These eight cones form the Jacobian J_{48} of the net $|B_{17}|$. Each of the lines OP' and OP'' is the conjugate of the other in I. They do not lie on K_9, nor are they basis for the net of cones $|B_{17}|$.

23. Images in (x') of the basis lines. If (x_1, x_2, x_3) are the line coordinates of OP_i, consider a neighboring line $(x_i + \epsilon\xi_i)$, and expand the expressions for x'_1, x'_2, x'_3 along this line in powers of ϵ. Since ϕ^2, $\phi\psi$, f, and their first derivatives vanish along OP_i, the expansions begin with ϵ^2. Divide by ϵ^2, eliminate the parameters ξ_1, ξ_2, ξ_3, and there results for the image of OP_i a quadric cone q'_i of (x').

These 8 quadric cones, $q'_1, \cdots, q'_8$, are components of 8 adjoint cubics of L'_6, the other component being the plane $x'_1 = 0$. Since the cubic adjoints have a tacnodal line at $x'_1 = x'_2 = 0$, the quadrics are tangent to $x'_1 = 0$ along this line. Their equations are of the form

---

* A. Wiman, "Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene," *Math. Annalen*, vol. 48 (1896), pp. 195-240.

$$ax'_1x'_3 + b\,x'^2_1 + cx'_1x'_2 + d\,x'^2_2 = 0.$$

The image of q'_i in (x) is composed of OP_i and the sextic cone σ_i; hence the quadrics must be tangent to L'_6 along all lines of intersection. Six intersections are at $x'_1 = x'_2 = 0$; each q'_i is tangent to L'_6 along three other lines.

If a similar expansion is carried out for the lines OP' and OP'', in both cases the equations reduce to $x'_1 = 0$. This plane, image in (x') of OP' and OP'', has all its intersections with L'_6 along the line $x'_1 = x'_2 = 0$.

One other line must be considered, OC, the ninth intersection of ϕ and ψ. Since ϕ and ψ are both invariant, this line is invariant. It does not lie on K_9 and is therefore an isolated invariant line.*

24. Now consider the general space transformation of degree n, defined by the equations

$$\begin{aligned} x'_1 &= p_{n-6}\phi^2 \\ x'_2 &= p_{n-6}\phi\psi \\ x'_3 &= p_{n-6}f \\ x'_4 &= u_{n-1}x_4 + u_n \end{aligned} \tag{9}$$

where p_{n-6}, u_{n-1}, u_n, are all ternary in x_1, x_2, x_3, and ϕ^2, $\phi\psi$, f, are the cones described above. As in previous cases, these equations define a (1, 2) correspondence between points of (x') and (x).

The basis elements are the 8 simple points $P_1, \cdots, P_8$, the points P' and P'' also simple, O^{n-1}, and a basis curve $\beta_{n(n-6)} : O^{(n-1)(n-6)}$, the intersection $(p_{n-6}, u_{n-1}x_4 + u_n)$. The variable part of the curve of intersection of two n-ics of the web is $c_{6n} : O^{6(n-1)}$. This curve lies on a sextic cone belonging to the net (7). Hence $p = 2$ for the curve c_{6n}, and the order of the surface of branch points L' is 6. This given another illustration of the violation of the Riemann-Roch theorem.

By reasoning identical with that used in the preceding cases, L'_6 is seen to be the cone appearing in the ternary transformation, and the surface of coincidences is the cone $K_9 : OP_1^3 \cdots OP_8^3$.

The correspondence between the two spaces is:

$$s_1 \sim s'_{6n}; \quad c_1 \sim c'_n; \quad s'_1 \sim s_n; \quad c'_1 \sim c_{6n}.$$

---

* M. Halphen, "Sur les courbes planes du sixième degré à neuf points doubles," *Bull. Société Math.*, vol. 10 (1882), pp. 162-172.

25. Images in (x') of the fundamental elements of (x). The points P_i are double on the surfaces representing x'_1, x'_2, x'_3 in equations (9), but simple on the monoid representing x'_4. If we expand these equations at a point in the neighborhood of P_i, in general the resulting image is the point O'. But all monoids of the web (9) have a common tangent plane at P_i, the plane tangent to $u_{n-1}x_4 + u_n = 0$ at this point. If P_i is approached on a line lying in this plane, the coefficient of ϵ in the expansion of x'_4 vanishes. Proceeding as we did in the ternary case, we obtain at once the equation of q'_i, the quadric cone already mentioned. Hence the images of the 8 points P_i are the 8 quadric cones q'_i. Every point on OP_i except O and P_i has for image the point O'; to the ∞^1 planes of approach to the line OP_i, correspond ∞^1 directions through O' on q'_i.

Similarly the image of the points P' and P'' is the plane $x'_1 = 0$. Each point of OP' and OP'' has O' as image and there is a (1, 1) correspondence between the pencil of planes through OP' (or OP'') and the pencil of lines through O' in $x'_1 = 0$.

The image of the curve $\beta_{n(n-6)}$ is a cone $R'_{6(n-6)}$. The equation of $R'_{6(n-6)}$ may be obtained by performing the ternary transformation upon the equation of the cone p_{n-6}.

The image of the point 0 is a fixed cone $Q'_{6(n-1)}$, since the image of any plane through O is composed of a variable cone Q'_6, and a fixed cone which must be of order $6(n-1)$.

The image of the tangent cone u_{n-1} is a curve $\xi_{n(n-1)}$, which lies on every s'_{6n}, and in particular, on the fixed cone $Q'_{6(n-1)}$. ξ' has O' to multiplicity $(n-1)(n-6)$. As was previously shown, the genus of ξ' is equal to $\frac{(n-2)(n-3)}{2}$, and this is also the genus of $Q'_{6(n-1)}$. Hence this cone has $\frac{5(n-1)(7n-10)}{2}$ double lines, of which $3(n-1)(3n-4)$ coincide at $x'_1 = x'_2 = 0$, and $\frac{(n-1)(17n-26)}{2}$ result from pairs of conjugate lines on u_{n-1}.

26. Equations of the involution. From the result of the Bertini ternary transformation, we have

$$x'_i = B_i(x_i) \qquad i = 1, 2, 3.$$

where B_i is of order 17, and also

$$\phi'^2 = J^2_{48}\phi^2$$
$$\phi'\psi' = J^2_{48}\phi\psi$$
$$f' = J^2_{48}f$$

where J_{48} is composed of the 8 sextic cones σ_i, conjugates of $P_1, \cdots, P_8$ in I, and enters to the second power since each P_i is double on ϕ^2, $\phi\psi$, and f. Using these results and solving in the usual way, we find the involution is defined by the equations

$$(10) \qquad \begin{aligned} x'_i &= p_{n-6}u_{n-1}(B_i)B_i \qquad i = 1, 2, 3. \\ x'_4 &= p_{n-6}(B_i)J^2_{48}(u_{n-1}x_4 + u_n) - p_{n-6}u_n(B_i). \end{aligned}$$

The order of the involution is $6(3n-1)$; its Jacobian is

$$J_I = u_{n-1}p_{n-6}(B_i)p^3_{n-6}[u_{n-1}(B_i)]^3 J^3_{48}.$$

The surfaces $s_{6(3n-1)}$ have $P_1, \cdots, P_8$ to multiplicity $6n-1$, and the lines $OP_1, \cdots, OP_8$ to multiplicity $6n-2$. P' and P'' are not basis in I. Other basis elements are: the curve $\beta_{n(n-6)}$; the lines $(p_{n-6}, p_{n-6}(B_i))$, of which $9(n-6)$ are invariant, and $(n-6)(17n-111)$ are arranged in pairs of conjugates; the lines (K_9, u_{n-1}); the lines (J_{48}, p_{n-6}) which are counted twice; the curve of intersection

$$[u_{n-1}(B_i),\ p_{n-6}(B_i)J^2_{48}(u_{n-1}x_4 + u_n) - p_{n-6}u_n(B_i)],$$

which after subtraction of the lines OP_i and the lines (K_9, u_{n-1}) is of order $3(n-1)(6n-5)$; call it $\gamma_{3(n-1)(6n-5)}$.

Now count the fixed elements of the intersection of two surfaces of the web $|\, s_{6(3n-1)} \,|$. In this case, there are $2(6n-2)$ lines consecutive to each OP_i which must be counted in the intersection. The total fixed intersection is of order $36(3n-1)^2 - 6(3n-1)$, so that the variable residual is a curve $c_{6(3n-1)}$.

27. Discussion of the involution. The conjugate of a plane s_1 through O is a cone B_{17}, meeting the plane in 17 lines through O. Nine of these lines are the intersection (s_1, K_9); the remaining 8 are arranged in pairs of conjugates. Hence every plane through O contains 4 pairs of conjugate lines.

As in previous cases, the conjugate of O in I is the composite cone $p_{n-6}u_{n-1}(B_i) = 0$. The conjugate of the cone u_{n-1} is a curve $\gamma_{3(n-1)(6n-5)} : O^{2(n-1)(9n-16)}$. This curve lies on the cone $u_{n-1}(B_i)$, and has each point P_i to multiplicity $6(n-1)$. It is basis for all surfaces $s_{6(3n-1)}$.

Let A be an arbitrary point on OP_i and let a line c_1 meet OP_i at A. The conjugate of P_i in I is known to be a sextic cone σ_i having OP_i triple, and each OP_k $(k \neq i)$ double. The variable conjugate of any plane through O is a cone of order 17 with OP_i sextuple; hence the conjugate of the plane

π through c_1 and OP_i is a cone Q_{11} passing fourfold through OP_k $(k \neq i)$ and triply through OP_i. Since an arbitrary line through O meets c_1 only once, an unrestricted generator of Q_{11} meets $\bar{c}$, the conjugate of c_1, once apart from O. The image of any point on σ_i, avoiding basis lines, is P_i; σ_i and Q_{11} intersect in 65 lines along basis lines and in a line $\bar{g}$, conjugate to g, which is the line through O consecutive to OP_i in the given plane. When the point generating c_1 passes through A, its conjugate describing $\bar{c}$ passes through $\bar{A}$ on $\bar{g}$, and since every point on $\bar{g}$ except O is conjugate to P_i, $\bar{A}$ is at O, that is, one branch of $\bar{c}$ at O has $\bar{g}$ for tangent. c_1 meets the cone $p_{n-6}u_{n-1}(B_i)$ in $12n-17$ variable points, which correspond to directions through O, in all, $12n-16$ passages through O on $\bar{c}$. But $\bar{c}$ is of order $12n-4$, since c_1 meets a surface of the web $|s_{6(3n-1)}|$ in $12n-4$ variable points. Hence $\bar{c}$ has an additional passage through O, giving O to multiplicity $12n-15$. The additional branch through O is also tangent to $\bar{g}$. For belonging to the net of Bertini sextics, there is a pencil $|\phi|$ all cones of which are tangent to the plane π at OP_i, and hence pass through $\bar{g}$. Consider the line c_1 as the limiting position of an arbitrary line c' not intersecting OP_i; c' has $18n-23$ intersections with $p_{n-6}u_{n-1}(B_i)$, corresponding to directions through O on its conjugate. It has also two intersections with any ϕ' of the pencil which approach coincidence at A as c' approaches c_1. Also of the $18n-23$ intersections with $p_{n-6}u_{n-1}(B_i)$, $6(n-1)$ approach coincidence at A. The latter correspond in the limit to the intersection (σ_i, u_{n-1}) and do not concern the proper image. The two points on ϕ, however, are on a self-conjugate surface; hence their conjugates must lie on ϕ, σ_i, and Q_{11}, that is on $\bar{g}$. This shows that OP_i plays a special role in the plane π; it is the only line whose conjugate meets the curve $\bar{c}$ in more than one point apart from O.

Just as in the case of the Geiser transformation, these lines OP_i are fundamental of the second kind.

The complete conjugate of a plane through O is a cone having each OP_i to multiplicity $6n$; a monoid of the web (10), however, contains these lines $6n-2$ fold. Since the intersection of two monoids can be reduced to the intersection of a cone and a monoid, these lines enter to multiplicity $6n(6n-2)$ in the intersection, and hence $2(6n-2)$ lines result from tangency. Of the tangent planes along these lines, 34 are given by J^2_{48}, and $6(n-6)$ by $p_{n-6}(B_i)$. Each plane tangent to J_{48} along OP_i has two branches of the surface tangent to it, and each branch of the cone $p_{n-6}(B_i)$ through OP_i has two consecutive lines common to all surfaces of the web $|s_{6(3n-1)}|$.

By methods identical with those used in the preceding cases, it is found that the variable curve $c_{6(3n-1)}$ has O to multiplicity $18n-23$, each P_i to multiplicity 6, meets $\gamma_{3(n-1)(6n-5)}$ in $n-1$ points apart from O, meets $\beta_{n(n-6)}$ in $17(n-6)$ points apart from O, and meets any monoid of the web $|\, s_{6(3n-1)} \,|$ in one variable point. All tangents to the curve at O lie on the same fixed cone.

28. Double curve. The double curve is the image of the plane curve $\delta_{3(6n-5)}$. Applying the methods of the preceding sections, its order is found to be $\dfrac{n(17n-9)}{2}$, and its genus $3(3n-4)(n-1)$.

The fundamental elements extraneous in the involution are: p^{6n-1}_{n-6} ϕ^{6n}, u_{n-1}. This result gives for the tangent cone u'_{6n-2} of the surface s'_{6n}, the composite cone whose equation is

$$x'^2_1 \prod_{i=1}^{8} q'^2_i R'_{6(n-6)} = 0.$$

V. *The Monoidal Involution Compounded with the Perspective Jonquières.*

29. Given a pencil of planes through OP, where O is the point (0001) and P is (0010),

$$x_1 + kx_2 = 0$$

and a pencil of cones of order m,

$$\phi_m + lx_2\psi_{m-1} = 0$$

where ϕ_m has OP to multiplicity $m-2$, and ψ_{m-1} has OP to multiplicity $m-3$. Through any line of O passes one plane of the first pencil and one cone of the second pencil; these intersect in a second line, not OP, which is the conjugate of the first in I.

Consider two spaces (x) and (x'). The equations

$$\begin{aligned} x'_2x_1 - x'_1x_2 &= 0 \\ x'_2\phi_m - x'_3x_2\psi_{m-1} &= 0 \end{aligned}$$

define a (1, 2) correspondence between the lines of O' and O. The transformation is given by the equations

$$\begin{aligned} x'_1 &= x_1\psi_{m-1} \\ x'_2 &= x_2\psi_{m-1} \\ x'_3 &= \phi_m \end{aligned} \tag{11}$$

and the image in (x) of any plane of (x') through O' is

$$ax_1\psi + bx_2\psi + c\phi = 0.$$

The basis lines are: OP to multiplicity $m-2$, and the $4m-6$ lines of intersection of ψ and ϕ as simple lines.

By using the fact that K is a component of the Jacobian of the net (11), its equation is found to be

$$\psi\phi_3 - \phi\psi_3 = 0 \tag{12}$$

where the subscripts indicate partial differentiation. It is of order $2m-2$ with OP to multiplicity $2m-4$, and the $4m-6$ simple basis lines OS_i.

An involution of rays through O exists in every plane through OP, its two intersections with K_{2m-2} being the double lines of the involution. The order of the involution is $2m-2$; a plane through O has for conjugate a cone J_{2m-2}.

The fact that the involution is of the same order as the coincidence cone is sufficient to show that it is a perspective Jonquières. The properties can also be derived directly.*

The conjugate of OP in I is a cone of order $2m-3$, say C_{2m-3}. It has OP to multiplicity $2m-4$, passes through the $4m-6$ lines OS_i, and is the first polar of K_{2m-2} with respect to OP. The plane through OP and any line OS_i is tangent to K_{2m-2}. Every line of such a plane is conjugate to OS_i, and since every plane through O meets each of these tangent planes in a line, every J_{2m-2} passes through OS_i, which is therefore basis in I. The J net is defined as follows: $J_{2m-2} : OP^{2m-3};\ OS_1 \cdots OS_{4m-6}$.

Any plane through O meets K_{2m-2} in $2m-2$ lines which must also lie on its conjugate J_{2m-2}. Since this is the total number of lines in which the plane meets J_{2m-2}, the class of the involution is zero.

30. Discussion of the fundamental elements of (x'). A plane of (x) through O corresponds to a rational cone of order m in (x'), Q'_m.

Every generator of ψ_{m-1} has for image in (x') the line $O'P'$, $x'_1 = x'_2 = 0$. This line is basis of order $m-1$ for the cones Q'_m, and it is the only fundamental line in (x').

L' is of order $2m-2$; the generators of L' and K are in (1, 1) correspondence. Since K meets ψ_{m-1} in $2m-4$ lines apart from basis lines, L' has $O'P'$ to multiplicity $2m-4$.

---

* P. Boyd, "On the Perspective Jonquières Involutions Associated with the (2, 1) Ternary Correspondence," *Amer. Journ. Math.*, vol. 34 (1912), pp. 291-324.

The $4m-6$ lines OS_i correspond to $4m-6$ fundamental planes in (x'), all of which pass through $O'P'$, and have one line of contact with L'. This accounts for all tangent planes which can be drawn through $O'P'$ to L'.

The image in (x') of the line OP is a cone $C'_{m-2} : O'P'^{m-3}$. The image of a plane through OP is a plane through $O'P'$, counted twice, since the order of the image is 2.

The intersection of L' and any cone Q'_m, apart from $O'P'$, consists of $2m-2$ lines of contact. The fundamental cone C'_{m-2} touches L' in $2m-4$ lines

31. The general transformation is given by the equations

$$\begin{aligned} x'_1 &= p_{n-m}x_1\psi_{m-1} \\ x'_2 &= p_{n-m}x_2\psi_{m-1} \\ x'_3 &= p_{n-m}\phi_m \\ x'_4 &= u_{n-1}x_4 + u_n \end{aligned} \tag{13}$$

where p_{n-m}, u_{n-1}, u_n are ternary in x_1, x_2, x_3, and ψ_{m-1}, ϕ_m, are the cones already discussed.

The basis elements are: O^{n-1}, the simple point $P \equiv (0010)$, $4m-6$ simple points S_i, and the curve $\beta_{n(n-m)}$ which is the intersection $(p_{n-m}, u_{n-1}x_4 + u_n)$, and has O to multiplicity $(n-1)(n-m)$. The variable curve of intersection of two surfaces of the web (13) is a curve $c_{mn} : O^{m(n-1)}$, P^{m-2}, lying on a cone of the net (11). The genus of c_{mn} is $m-2$; hence the order L' is $2m-2$. It is the cone already considered, and as before the locus K is given by equation (12). As in the Bertini case, the Riemann-Roch theorem does not hold.

The correspondence is therefore:

$$s_1 \sim s'_{mn}; \quad c_1 \sim c'_n; \quad s'_1 \sim s_n; \quad c'_1 \sim c_{mn}.$$

32. Images in (x') of fundamental elements of (x). Any plane through O meets c_{mn} in m points apart from O. Hence a line of (x') meets the image of the plane in m points. This image is a cone Q'_m and the image of O is a fixed cone $Q'_{m(n-1)}$. If in addition the plane through O passes through P, the image is a single plane, counted twice, passing through $O'P'$; hence the image of P is a cone of order $m-2$, C'_{m-2}. This cone has the line $O'P'$ to multiplicity $m-3$, for the image of $O'P'$ in (x) is known to be $\psi_{m-1} : \overline{OP}^{m-3}$.

If a plane of (x) passes through OP and a point S_i, it is one of the $4m-6$ tangent planes to K_{2m-2}. As before, the complete image in (x') is composed of $Q'_{m(n-1)}$, C'_{m-2}, and a tangent plane through $O'P'$ to L' counted twice, once on account of the tangent plane to K_{2m-2}, and once as the image of S_i.

All points on the lines OS_i and OP have their images at O', and there is a (1, 1) correspondence between the planes through OS_i and the pencil of rays through O' in the image plane of S_i; also between the planes through OP and the ∞^1 generators of the cone C'_{m-2}.

The image of $\beta_{n(n-m)}$ is a cone $R'_{m(n-m)}$ which has $O'P'$ to multiplicity $(m-1)(n-m)$.

The image of the tangent cone u_{n-1} is a curve $\xi'_{n(n-1)}$, with O to multiplicity $(n-1)(n-m)$ and P to multiplicity $(m-1)(n-1)$. ξ' is basis or every surface s'_{mn}, and in particular lies on the fixed cone $Q'_{m(n-1)}$, no generator of which can meet ξ' in more than one point. Hence $Q'_{m(n-1)}$ contains $O'P'$ to multiplicity $(m-1)(n-1)$. The genus of $Q'_{m(n-1)}$ is $\dfrac{(n-2)(n-3)}{2}$; it has $(m-1)(n-1)(n-2)$ double lines resulting from pairs of conjugates on u_{n-1} and $\dfrac{(mn-m-n+1)(mn-m-n)}{2}$ others at $O'P'$.

33. Equations of the involution. From the ternary transformation we know that

$$x'_1\psi'_{m-1} = x_1\psi_{m-1}T_{4m-6}C^{m-2}_{2m-3}$$

and similar expressions for $x'_2\psi'_{m-1}$, and ϕ_m, where T is composed of the $4m-6$ fundamental planes and C_{2m-3} is the fundamental cone, conjugate of P. Solving in the usual way, we obtain

$$\begin{aligned} (14) \qquad & x'_1 = p_{n-m}u_{n-1}(J_i)J_i \qquad i = 1,\ 2,\ 3. \\ & x'_4 = p_{n-m}(J_i)T_{4m-6}C^{m-2}_{2m-3}\ (u_{n-1}x_4 + u_n) - p_{n-m}u_n(J_i). \end{aligned}$$

The involution is of order $2mn-m-n$. All surfaces of the web (14) have O to multiplicity $2mn-m-n-1$, P to multiplicity $2mn-3n-m+3$, and each S_i to multiplicity n. The line OP has multiplicity $2mn-3n-m+2$, and each line OS_i has multiplicity $n-1$. Two monoids intersect in $n-1$ additional lines along each line OS_i because of tangency, and in $(m-2)$ $(2mn-3n-m+2)$ additional lines along OP for the same reason.

Other basis elements enter as in the preceding cases. They are: the curve $\beta_{n(n-m)}$, lines $(p_{n-m}, p_{n-m}(J_i))$, lines (p_{n-m}, T_{4m-6}), lines $(p_{n-m}, C^{m-2}_{2m-3})$ lines (K_{2m-2}, u_{n-1}) and a curve γ of order $(n-1)(2mn-3m-n+2)$. The residual intersection, conjugate of a line of (x), is a curve $C_{2mn-m-n}$.

34. Discussion of the involution. The conjugate of a plane through O is a cone of order $2m-2$. The plane and cone meet in $2m-2$ lines, all of which are invariant, since K is of order $2m-2$. Hence the class of the involution is zero. If the plane passes through OP, it contains two invariant lines, and all other lines in the plane are arranged in pairs of conjugates. The conjugate of such a plane is itself, together with the cone C_{2m-3} which is the conjugate of P in the involution.

Consider any line c_1 in a plane π through OP, intersecting OP at A. Since the conjugate of this plane is itself, the conjugate of c_1 is a plane curve $\bar{c}$. c_1 meets a surface of the web $| s_{2mn-m-n} |$ in $2n-2$ variable points; this is the order of $\bar{c}$. An arbitrary line of the pencil O meets c_1 once, and hence meets $\bar{c}$ once apart from O. Let $\bar{g}$ be the line conjugate to g, where g is consecutive to OP in π; $\bar{g}$ lies on C_{2m-3}. All other intersections of π and C_{2m-3} are at OP. Consider the pencil of cones of the net (11) which passes through g and $\bar{g}$, and let c_1 be the limiting position of a line c' which does not meet OP. c' intersects the cones of this pencil in $m-2$ points which approach coincidence at A. The conjugate of c' has $2mn-3m-n+2$ passages through O, given by the intersections of c' with $p_{n-m}u_{n-1}(J_i)$; of these intersections $2mn-2m-3n+3$ approach coincidence at A. In the limit, the conjugates of these latter points lie on u_{n-1} and C_{2m-3}; they are extraneous for the proper image of c_1. The conjugates of $m-2$ points, however, which lie on a self conjugate cone, must be on the same cone, on π, and on C_{2m-3}, that is they must lie on $\bar{g}$. But the conjugate of every point on $\bar{g}$ except O is at P; hence $\bar{c}$ has $m-2$ passages through O at $\bar{g}$, and $2n-m-1$ others, a total of $2n-3$. As in the Bertini case, the line OP plays a special role in the plane; it is the only line whose conjugate meets $\bar{c}$ in more than one point apart from O.

If we carry through a similar procedure with a line c_1 lying in any plane π through OS_i, there is no difficulty in showing that the conjugate curve $\bar{c}$ lies on a cone Q_{2m-3}, each generator of which meets $\bar{c}$ once. $\bar{c}$ has O to multiplicity $2mn-2n-3m+4$, because of $2mn-2n-3m+3$ variable intersections of c_1 with $p_{n-m}u_{n-1}(J_i)$, and one additional passage on $\bar{g}$, the conjugate of the line g consecutive to OS_i in π. The order of $\bar{c}$ is $2mn-2n-m+1$.

The lines OP and OS_i are fundamental of the second kind in I. The conjugate of any point on OP is the line OP to multiplicity $2mn-3n-m+2$, and the conjugate of any point on OS_i is OS_i to multiplicity $n-1$.

The conjugate of O in I is the composite cone $p_{n-m}u_{n-1}(J_i)=0$. The conjugate of u_{n-1} is the curve γ basis for the web $| s_{2mn-m-n} |$. It lies on

$u_{n-1}(J_i)$, each generator of which meets it in one point apart from O, has P to multiplicity $2mn-2m-3n+3$, and each S_i to multiplicity $n-1$.

It has already been noted that there are $n-1$ intersections of two monoids of the web (14) along each line OS_i due to tangency, and $(m-2)$ $(2mn-m-3n+2)$ along OP for the same reason. Along each OS_i, $n-m$ of the consecutive lines lie in the tangent planes to $p_{n-m}(J_i)$, one is in the plane OPS_i and $m-2$ in the tangent plane to C_{2m-3}, taken $m-2$ times. Along OP, tangency is more complicated. Each branch of C_{2m-3} contains $m-2$ lines common to all $s_{2mn-m-n}$, and since C_{2m-3} enters to multiplicity $m-2$, each branch must be counted that number of times. Each plane OPS_i has $m-2$ lines consecutive to OP which are common to all the monoids of I. Finally, each of the $(2m-3)(n-m)$ branches of $p_{n-m}(J_i)$ has $m-2$ lines consecutive to OP and common to the monoids of I. This accounts for all intersections due to contact.

The variable curve $C_{2mn-m-n}$, conjugate of a line of (x), has O to multiplicity $2mn-3m-n+2$, P to multiplicity $2m-3$, and each S_i as simple point. It meets γ in $n-1$ points apart from O, and meets any monoid of the web in one variable point. All tangents to the curve at O lie on a fixed cone.

35. Double curve. The intersection $(s_1, s_{2mn-m-n})$ is composed of (s_1, K_{2m-2}) and a residual $\delta_{2mn-3m-n+2}$. The image of δ in (x') is a double curve δ' of order $n(n-1)(m-1)$. By applying Zeuthen's formula as in previous cases, the genus of δ' may be calculated. It is $\dfrac{(n-3)(2mn-2m-n)}{2}$.

Every surface s'_{mn} has $O'P'$ as a multiple line of order $n(m-1)$, for it may be shown algebraically that a line through P' meets s'_{mn} in n variable points. Since the image of $O'P'$ is ψ_{m-1}, every s'_{mn} on transformation has an extraneous factor $\psi_{m-1}^{n(m-1)}$. As in previous cases, u_{n-1} and p_{n-m}^{mn-1} are also extraneous. The remaining component is of order $2mn-m-n+1$, consisting of a plane of (x) and its conjugate $s_{2mn-m-n}$.

The tangent cone to s'_{mn} at O' is of order $mn-2$. It consists of $R'_{m(n-m)}$, C'^{m-2}_{m-2}, and $4m-6$ planes, images of the $4m-6$ points S_i.

Self-Projective Rational Septimics.

BY R. M. WINGER.

§ 1. *Introduction.*

In two former papers* the different species of self-projective rational curves as far as the sixth order were obtained. According to the criterion of classification adopted, there were found two species of cubics, eight of quartics, twelve of quintics and twenty-six of sextics.

The notation and nomenclature of the earlier papers are retained as far as possible,—in particular *type* is employed in the sense defined in the second paper (§ 1). We first seek the types of rational septimics, then describe those properties which the method so readily reveals. A major objective is the analysis of the higher singularities which occur in great variety and considerable complexity. For one of the most interesting features of the study of self-projective curves is to see how the singularities adjust themselves to the demands of the groups. The possibility of such adjustment is of course an important factor in the existence of curves associated with any group while the number of species depends on the variety of ways in which the adjustment can be effected.

Since rational curves occur in conjugate pairs,† to every self-projective plane rational curve ρ^m, $m > 5$, corresponds a self-projective rational curve ρ^m in a space S_{m-3} of $m - 3$ dimensions. And a classification of plane curves implies a classification of conjugate curves. The equations of the latter can be written down at once since sections by S_{m-4}'s of the curve in S_{m-3} are simply sets in the fundamental involution of the plane curve. Reciprocally, the line sections of the plane curve are sets in the fundamental involution of the conjugate. For example we have as a pair of conjugate septimics admitting a dihedral G_{14},—one in the plane and the other in S_4:

$$x_0 = t^6, \qquad x_1 = t, \qquad x_2 = t^7 + 1,$$

$$x_0 = t^5, \qquad x_1 = t^4, \qquad x_2 = t^3, \qquad x_3 = t^2, \qquad x_4 = t^7 + 1.$$

---

* "Self-Projective Rational Curves of the Fourth and Fifth Orders," *American Journal of Mathematics*, January, 1914 (referred to in this paper as Q); and "Self-Projective Rational Sextics," *Ibid.*, January, 1916, referred to as S.

† Conner, *Trans. Amer. Math. Soc.*, April, 1912, p. 265. When $m = 5$ we have two conjugate plane quintics.

Again the dual of any self-projective curve is obviously a self-projective curve whose order is the class of the original. Thus the projective lemniscate

$$x_0 = t, \qquad x_1 = t^3, \qquad x_2 = t^4 + 1$$

is invariant under the octahedral G_{24} (Q, p. 61). The dual is the projective astroid, $\xi_i = x_i$, the point equations of which are

$$x_0 = t^6 - 3t^2, \qquad x_1 = -3t^4 + 1, \qquad x_2 = 2t^3.$$

This is one type of octahedral sextic (S, p. 47, No. 25).

In this manner we are able to predict an icosahedral rational decimic and can write its equations from the icosahedral rational sextic (S, p. 47, No. 26).* The method is however of limited application since the dual curve is not usually the only, nor the most general curve of a given order admitting a given group.

Several general theorems for rational curves of any order were enunciated in the earlier papers. The following generalization of a theorem stated for the quintic (Q, p. 67) is fundamental in the study of the rational septimic or indeed any odd curve:

A rational curve ρ^m of odd order cannot admit the icosahedral, the octahedral or the tetrahedral group, or a dihedral G_{2n}, n even.

For the binary group g_n must leave the flex form unaltered, since obviously a flex can go only into a flex. Now the flex form is of order $3\ (m - 2)$, an odd number, while all the invariant sets of the groups in question, particular as well as general, are even.† Hence a non-singular flex form could not break up into sets conjugate under the group. Moreover if the flex form have multiple roots, they must occur an even number of times,—leaving an odd number of simple flexes which precludes the existence of the groups as before.

Combined with a former theorem (Q, § 10) *this limits the types of odd curves to those invariant under cyclic groups of orders 2, 3, · · ·, m, and dihedral groups of orders 2n, n odd and $\leq m$.*

In spite of this severe restriction, there are found to exist 25 varieties of projectively distinct rational septimics many of them with arbitrary constants remaining. The appended table exhibits in canonical form the parametric equations of these types, together with the generating binary transformations of the characteristic groups. The transformations for the dihedral groups are to be combined with $t' = 1/t$. The multipliers, except a which is an arbitrary constant, are primitive roots of unity where

---

* Cf. Klein, *Ikosaeder*, pp. 218-219 and Winger, *Mathematische Annalen*, Band 93 (1925), p. 213.

† Thus the special sets of the tetrahedral, octahedral and isocahedral binary groups correspond to (a) the vertices, (b) the centers of the faces and (c) the midpoints of the edges of the associated regular bodies and contain therefore an even number of points. Again a dihedral group g_{2n} in the binary domain has but three special sets—a set of two and two sets of n parameters.

$$\omega^3 = i^4 = \eta^5 = \epsilon^7 = 1.$$

The reference triangle is in every instance an invariant triangle of the group. Cyclic and dihedral groups of order k are designated by C_k and D_k respectively.

For convenience in analyzing singularities we collect here certain useful data. The ρ^7 is of class 12, has 15 double points, 15 flexes and 40 double lines. These numbers are modified of course in accordance with Plücker's equations when cusps occur. Thus for the rational curve of order n with r cusps, the number of double lines δ is

$$\delta = 2(n-2)(n-3) - \tfrac{1}{2}r(4n-r-11)$$

or when $n = 7$, $$\delta = 40 - \tfrac{1}{2}r(17-r).$$

A tangent with k-point contact (not at a multiple point absorbs $k-2$ flexes and $\frac{1}{2}(k-2)(k-3)$ double tangents. From such a point can be drawn only $\nu - k$ simple tangents, where ν is the class of the curve.

Two standard methods are available for the analysis of higher singularities—expansion in the neighborhood of the singularity and birational transformation. But both require the ternary equation of the curve. Other methods are desirable for rational curves which are normally given parametrically. Various devices are useful. Most important of all here is the requirements of the groups. Thus under binary cyclic groups of order n we have one set of two parameters each of which is fixed while all other parameters are made up of sets of n. Likewise under binary dihedral groups there are three special sets (footnote, p. 2) while all other parameters fall into general sets of $2n$ points. The number of cusps can be ascertained from the reduction in class and the location of the cusps can be determined from the flex equation which is readily calculated.*

The main difficulty arises in enumerating the nodes and bitangents which go to make up a compound singularity although the number of the former can sometimes be read directly from the parametric equations. In this connection the notion of *excess* developed by Dr. Scott[*] is of great assistance. Thus for simplicity consider a multiple point with a single tangent. If every line on the points cuts out k coincident parameters, the point is said to be of *order* k. Likewise if the tangent at the multiple point counts for l common lines of the curve and any point on the tangent; the multiple point is of *class* l. The number of double points d (including cusps) contained in the singularity is

---

* Hilton, *Plane Algebraic Curves*, p. 138.

$$d = k(k-1)/2 + X$$

where X represents the *excess* over the normal—or the number of *latent* double points. Then it is proved that the number of double lines δ (including inflexional tangents) is

$$\delta = l(l-1)/2 + X$$

i. e. *the excess of double lines is equal to the excess of double points.*

For example, curve (5) (q. v.) has at 0 a triple point with coincident parameters and a 7-point contact tangent, i. e. the triple point is of order 3. The line equations show that this point is of class 4, hence

$$d = 3 + X, \qquad \delta = 6 + X.$$

It is found that two of the d double points are cusps and that $X = 3$. Hence $\delta = 9$ but the flex equation shows that three of the double lines are flexes. Thus the singularity is equivalent to 2 cusps, 4 ordinary double points, 3 flexes and 6 bitangents.[†] *These numbers as always are in addition to the flexes and bitangents absorbed by the two cusps.*

If the multiple point has more than one tangent, the class of each branch is ascertained separately, but the excess is reckoned for the singularity as a whole.

The number of the double points in a higher singularity can also be determined by writing out the double point equation.[‡] But this equation for the rational septimic is a determinant of order 6 whose elements are binary quintics so that the expansion is tedious except in very special cases.

---

[*] "The Nature and Effect of Singularities of Plane Algebraic Curves," *American Journal of Mathematics*, Vol. XV (1893), pp. 230-237.

[†] We shall commonly use "double line" as synonymous with bitangent, and "double point" for the dual singularity, though strictly the double line as a line singularity includes the flex tangent as a special case.

[‡] Clebsch, *Vorlesungen über Geometrie*, (1876), p. 889.

§ 2. *Rational Septimics of Maximum Symmetry.*

When referred to absolute coordinates (as explained Q, § 4) dihedral groups of order $2n$ imply symmetry about n equi-spaced axes. When n is odd, as must be the case for curves of odd order, the axes are conjugate under the group, i. e. the symmetry is the same with respect to each axis. The maximum symmetry is attained when $n = m$, the order of the curve. In view of the formulas for dihedral curves (Q, p. 66) and a theorem of the previous section we may say: *There is always at least one rational curve of odd order m which possesses m-fold symmetry, viz., symmetry with respect to m equi-spaced lines about a point.*

There are three types of rational septimics with this maximum symmetry—curves 23, 24, 25. These curves illustrate admirably the progressive specialization and concentration of singularities to conform to the requirements of the group. For while none of the curves has a non-vanishing absolute invariant, each is of a different degree of specialization. To consider only the double points, some of them must lie on the axes and of course they must be distributed symmetrically. Hence the possibilities are (1) two on each axis and an additional one at the intersection of the aves,[*] (2) one on each axis and 8 coincident at the origin and (3) all coincident at the origin. In a similar manner the flexes and double lines display a variety of coincidences.

We shall now discuss the three curves in detail, noting first the properties common to all. *Each has 7 flexes lying on a line, namely the centers of reflexion, $t^7 + 1$, cut out by x_2.* These flexes are a special set under both the binary and the ternary group. The other special set of seven points conjugate under both groups are the contacts $t^7 - 1$ of simple tangents from the centers, one lying on each axis. The multiple point at the origin, while different for each curve, has but two parameters which are the special set of two under the binary group. The multiple point itself is the unique fixed point of the ternary group. The pencil of invariant conics, $x_0x_1 + \lambda x_2^2 = 0$, all touch the tangents of the special multiple point where they cut the line of centers.

Curve 23. Each branch of the double point which lies at the origin has a tangent with 6-point contact. These nodal tangents thus absorb 8 flexes and 12 double lines. The remaining double points lie by twos on the axes of reflexion and the remaining double lines meet by fours at the centers.

---

[*] We shall call this point, the sole fixed point of the ternary group, the *origin* since it corresponds to the origin in the metrical representation by absolute coordinates.

The parameters of these 14 double points are the 35th roots of 1 exclusive of the 7th roots.

Noteworthy among the invariant conics are (1) two each on seven double points, (2) two each on seven pairs of tangents at the same, (3) four each on seven double lines, (4) four each on fourteen contacts of double lines, (5) four each on the fourteen ordinary intersections of seven double lines, (6) four each on fourteen tangents at the points of (5), (7) one with seven contacts, (8) one on seven flex tangents. The nodal tangents x_0 and x_1 each count as five common lines of the conics and ρ^7. The remaining fourteen normally constitute a set of general conjugate lines.

Curve 24. The multiple point is formed by the union of two cusps of higher order, but with distinct tangents, each of which is equivalent to one double point, one cusp, two flexes and two double lines. Four flexes and 15 double lines were previously required to make the cusps, while the coincidence of the cusps engulfs four new double points. One double point lies on each axis and three double lines meet at each center, which accounts for the remaining double elements. The parameters of the seven ordinary nodes are the 21st roots of unity exclusive of the 7th roots.

The system of invariant conics include in particular: (1) three each on seven double lines, (2) three each on fourteen contacts of double lines, (3) three each on the fourteen ordinary intersections of seven double lines, (4) three each on the tangents at the points of (4), (5) one on seven double points, (6) one on seven pairs of tangents at the same, (7) one on seven flex lines and (8) one with seven contacts. The tangents at the four-fold point each count for three common lines of the conics and the septimic. There remain fourteen common lines which form a general conjugate set under the groups except in such cases as (7) and (8) when the common lines are special sets of seven conjugate lines. But for conics (1) the common lines are the seven double lines which are a set of seven under the ternary group while the contacts are a set of fourteen under the binary.

Curve 25. The multiple point at the origin is a six-fold point, arising from the coincidence of two special triple points each of which is formed from two cusps and one double point. The singularity is thus equivalent to 11 double points and 4 cusps, the latter absorbing in addition 8 flexes and 26 double lines. It has however but two distinct parameters and presents the appearance of an ordinary double point. Two double lines meet at each center which accounts for all that remain.

The invariant conics of note are: (1) two each on seven double lines, (2) two each on fourteeen contacts of double lines, (3) two each on fourteen ordinary intersections of seven double lines, (4) two each on fourteen tangents at points in (3), (5) one on seven flex tangents, and (6) one with seven contacts. Each tangent of the multiple point counts for one common line of conics and the base curve. Since the curve is of class eight, 14 common lines remain,—a set of 14 conjugate lines in general.

§ 3. *Other Rational Septimics with Multiple Symmetry.*

Ternary cyclic groups of even order have one element of period two. The axis of this reflexion will, if the invariant curve be taken in appropriate metrical form, be an axis of symmetry. Or the center of the reflexion, when taken for the origin, will be a point of symmetry, i. e. a *center* of the curve. Aside from these cases the only other rational septimics with symmetry are the curves with a dihedral G_6 or G_{10}. In metrical form the first has 3-fold and the second 5-fold symmetry.

Curve 21. The three centers of reflexion are flexes, t^3+1, and lie on a double line, x_2. The contacts, 0, ∞, of the double line are the special set of two under the binary g_6. The other special sets are the flexes and the contacts, t^3-1, of simple tangents from the flexes, one on each axis. The other intersections of each axis are made up of three double points. Three double lines meet at each center—in addition to the double line common to the three centers. There remain 6 double points, 12 flexes and 30 double lines.

The invariant conics pass through the contacts of the special double line and cut out besides two sets of conjugate points in general. Notable conics of the system are: (1) two each on six flexes, (2) two each on six flex tangents, (3) three each on three double points, (4) three each on six pairs of tangents of double points, (5) three each on three double lines, (6) three each on three pairs of contacts of double lines, (7) three each on three pairs of ordinary intersections of double lines in (6), (8) three each on tangents at three pairs of points in (7), (9) five each on six double lines, (10) ten each on six contacts of double lines (9) (one contact of each), (11) one on three flex tangents, (12) one on six double points, (13) one with three contacts.

Curve 22. x_2 cuts out five flexes, t^5+1, which are the centers of reflexion, and two other points, 0, ∞, which are the special set of two under the binary group. The other special set of five under g_{10} are the contacts,

t^5-1, of the simple tangents from the centers, one on each axis. Three double points lie on each axis which accounts for all the double points. Four double lines meet at each center, leaving 20 unaccounted for.

The invariant conics all have contact at 0 and ∞, cutting out besides sets of ten conjugate points. Special conics of the pencil are: (1) three each on five double points, (2) three each on five pairs of tangents at double points, (3) four each on five double lines, (4) four each on five pairs of contacts of the same, (5) four each on five pairs of ordinary intersections of the same, (6) five each on five pairs of tangents at these extra intersections, (7) two each on ten double lines, (8) four each on ten contacts of double lines in (7) (one on each line), (9) one on ten flexes, (10) one on ten flex tangents, (11) one on five flex tangents (t^5+1), (12) one with five additional contacts, i. e. with seven contacts.

The flex form is

$$(t^5+1)[at^{10}+(6a^2-25a+21)t^5+a].$$

The condition that this have a pair (and hence five pairs) of equal roots is

$$(a-1)(2a-3)(2a-7)(3a-7)=0.$$

If $a=1$, the curve is degenerate. But the other values of a lead to special curves of interest. Thus

If	*The curve has*
$2a=3$	undulations at t^5-1
$2a=7$	5-point contact tangents at t^5+1
$3a=7$	cusps at t^5-1.

The last curve is self-dual, admitting besides the collineation G_{10} the polarity *

$$\xi_0=x_1,\quad \xi_1=x_0,\quad \xi_2=x_2,$$

which has the fixed conic $2x_0x_1+x_2^2=0$. The product of this with the ten collineations gives ten correlations, six of which are polarities while the others are of period ten. We thus obtain a G_{20} containing ten collineations (including the identity) and ten correlations.

The curve has 5 cusps, 5 flexes, 10 double points and 10 double lines. Each axis is tangent at one of the cusps while each center carries two double lines.

---

* We take the equation of the curve, as is possible, in the form

$$x_0=3t^7+7t^2,\ x_1=7t^5+3,\ x_2=\sqrt{42}(t^6+t).$$

These three curves will differ somewhat in properties from the general case. In particular the list of special invariant conics would require revision, a revision however which is easily accomplished.

§ 4. *The Cyclic Septimics, $n < 7$.*

We shall now take up a systematic discussion of the other types of curves. While these lack the beautiful symmetry of the dihedral curves, they are not without interest. On the whole the cyclic groups require greater concentration of singularities than do the dihedral, except for those curves whose specialization lead to dihedral cases, as e. g. (2) and (6). And it is in the cyclic curves that the problem of the singularities is most difficult.

There are two types of ternary cyclic groups G_n for all values of $n > 2$, viz. those generated by (a) the homologies and (b) the orthogonal collineation.* When $n < 6$ these are the only two types. When $n = 6$ we have an additional type with a single invariant triangle, namely that whose cyclic subgroup of order three is generated by a homology. The multipliers may be taken as $1, -\omega^2, -1$. When $n = 7$ we have again a third type, viz., a group with a single fixed triangle but without an invariant conic. Its multipliers are $1, \epsilon, \epsilon^3, \epsilon^7 = 1$.

Curve 1. The curve admits a reflexion of which ξ_0 is the center and x_0 is the axis.† The center lies on the curve and is a point of inflexion. The axis cuts out three double points and the contact of the simple tangent from the center. The other tangents from the center must be double lines—four in number. The remaining 14 flexes and 12 double points are harmonically perspective in pairs from the center.

Curve 2. The group is a cyclic G_3 generated by an orthogonal collineation. The fixed triangle consists of a double line, whose contacts are the fixed points of the binary group, and two lines on the contacts.

Curve 3. The group is generated by a homology of which ξ_2 is center and x_2 is axis. The center is a point of undulation and the axis cuts out two

---

* The orthogonal collineation can be written

$$x'_0 = \epsilon x_0, \qquad x'_1 = \epsilon^{-1} x_1, \qquad x'_2 = x_2, \qquad \epsilon^n = 1.$$

This is characterized by having an invariant conic, as well as a single invariant triangle. The invariant cyclic subgroup of the dihedral group is generated by an orthogonal collineation.

The multipliers of the homology may be taken as $\epsilon, 1, 1$ or $\epsilon, \epsilon, 1$.

† The vertex of the reference triangle opposite x_i we shall designate by ξ_i.

triple points and a flex. This flex and undulation are the fixed points of the binary group. All sets of conjugate points lie on lines through the center and these lines in general cut out two sets of conjugate points. There are however some exceptions: x_0 is the undulation tangent and meets therefore in only three additional points and x_1 is a flex tangent with contact on the axis; two triple tangents meet at the center. And the lines joining the center to the triple points cut out single sets of conjugate points under the ternary group though these lines present no exception for the binary group since the triple point parameters are sets under g_3. There remain 9 double points, 12 flexes and 33 double lines. The flex form is

$$-t^{12} + (2c - 5b)t^9 + (14 + 5ab - 8bc)t^6 + 7b - abc)t^3 + 2ab.$$

The double points likewise the flexes must lie in threes on lines through the center. The double lines must fall into sets of three meeting on the axis such that the contacts of a set form two sets of points on lines of the center.

Curve 4. The group is a cyclic G_4 generated by an orthogonal transformation. The center of the reflexion ξ_2 is a flex and the axis x_2 is a cusp tangent which cuts out besides two double points. The other sides of the fixed triangle are the flex tangent and the line joining the flex and the cusp. The cusp and flex parameters are the fixed points of the binary group. Four double lines meet at the center. There remain 12 double points, 12 flexes and 28 double lines (the cusp having reduced the number by 8). The flexes and double points and the contacts of the double lines must be harmonically perspective in pairs from the center. There is a pencil of invariant conics, $x_0x_1 + kx_2^2 = 0$, each of which in general cuts out three sets of conjugate points in addition to the cusp.

Curve 5. The group is again a cyclic G_4 but is generated by a homology. We have in fact both a reflexion and a homology (of period four) with a common center and axis. The axis x_2 cuts out an ordinary four-fold point, equivalent to 6 double points, and a triple point with coincident parameters, equivalent to two cusps and one double point. The center ξ_2 is a special triple point with coincident parameters and a single tangent x_0 which has 7-point contact. All fixed points under the group are now accounted for so that all others must occur in sets of four. The triple point ξ_2 certainly counts for three of the remaining 6 double points. Hence it must absorb them all, otherwise we should be left with a set of points fewer than four. The excess of the point is thus 3. In forming the line equations, the factor t^2 drops out so that two cusps are included in the singularity which is thus equivalent to 2 cusps, 4 double points, 3 flexes and 6 double lines. (See end of § 1.) There remain 8 double lines, 26 having been used in the formation of the 4 cusps, and 4 flexes.

All sets of four conjugate points lie on lines through the center. Among them may be mentioned (a) the four flexes $t^4 - 7$, (b) the contacts of tangents from the triple point ξ_0 and (c) the contacts of tangents from ξ_1. The 8 double lines must fall into two quadruples, the lines of each set meeting in a point of the axis and the contacts forming two sets of conjugate points.

Curve 6. The group is generated by an orthogonal transformation, the fixed triangle of which is composed of two tangents and their chord of contact. The invariant conics $x_0x_1 + kx_2^2 = 0$ are tangent at the fixed points of the binary group and cut out besides two sets of conjugate points. The curve exhibits no striking geometric features or specialization, all its points,—except the two fixed points already noted,—as well as their parameters occurring in sets of five.

Curve 7. ξ_0 (parameter ∞) is an ordinary cusp and ξ_1 (parameter 0) is a cusp of higher order having a 4-point contact tangent, x_2 being the common tangent of the two. These are the only fixed points and parameters under the ternary and binary group. The second singularity would appear to be a ramphoid cusp, which is equivalent to one double point and one cusp, but we should then be left with 12 double points which is impossible under a G_5. The most natural explanation is that in some way ξ_1 absorbs four double points. The flex equation and the class indicate that one cusp and one flex are included in the singularity. Now a special case of the curve having the same singularity, at 0 is

$$x_0 = t^7 + t^2, \quad x_1 = t^4, \quad x_2 = 1.$$

The double point equation of this curve is $t^8(t^{10} + 2t^5 + 2) = 0$ which says that there are four double points at 0 and six at ∞. We conclude therefore that the singularity in question counts for three double points and one cusp, hence the excess is 3. Accordingly three additional double lines (bitangents) are absorbed. Finally the two cusps reduce the number of double lines by 15. We have thus accounted for 5 double points, 5 flexes and 20 double lines. The invariant conics are $x_0^2 + kx_1x_2 = 0$, each of which cuts four times at ξ_1 while the remaining intersections form two sets of conjugate points in general.

Curve 8. The group is generated by a homology with ξ_0 as center and x_0 as axis. The former is a cusp (parameter ∞) with a 7-point contact tangent. ξ_2 is a cusp which has a 5-point contact tangent passing through the other cusp. The axis contains the second cusp and an ordinary 5-fold point—equivalent to 10 double points. The flex form is $t^4(7t^5-3)$ so that four flexes fall at 0 and six at ∞. The cusp 0 is of order 1 and class 3. The tangent thus counts for three double lines—one bitangent and two flexes. But the singularity contains one latent double point, accordingly an additional double line (a bitangent) is absorbed. Again the cusp ∞ is of order 1 and class 5, hence the tangent is equivalent to 10 double lines—consisting of 4 flex lines and 6 bitangents. But as the singularity contains two latent double points, its excess is 2 and two additional double lines (both bitangents) are absorbed. Finally the two cusps use up 15 bitangents. We have thus accounted for all the double points, 10 flexes and 25 double lines.

Sets of conjugate points lie on lines through the center. Among these are (a) the five remaining flexes, (b) the contacts of tangents from the other cusp, (c) the five contacts of tangents from the vertex ξ_1. Indeed the ten contacts of tangents from an arbitrary point of x_0 break up into sets of five conjugate points. For the coordinates of such a point may be taken as $(0, 1, k)$ when the contacts of tangents are given by $2kt^{10}-(3k+7)t^5-2=0$. When $k=0$, ∞ we get the two sets of five already noted. When $k=-1$ or $-49/9$, i. e. when the point is at the 5-fold point or the intersection of the flex tangents, the two sets reduce to one repeated. Incidentally this proves, as is otherwise evident, that the flex tangents meet in a point of the axis. Similarly, the 15 double lines must meet in sets of five points on the axis while the contacts of each quintet fall on two lines of the center which carry five each.

Curve 9. Since the group is a cyclic G_6, there are two subgroups—one of order two and the other a cyclic G_3. The latter is generated by an orthogonal transformation and has the same fixed triangle as the whole group. x_2 is a multiple line that has 5-point contact at ∞ (ξ_0) and simple contact at 0 (ξ_1) and which counts for 3 flexes and 7 double lines. The axis of reflexion, x_0, cuts out three double points while from the center ξ_0, run three double lines—with contacts $2t^6-1$. The flex form is

$$t^3[10t^{12}+(10a-35)t^6+a].$$

The cyclic subgroup of order 3 has three pencils of invariant conics, $x_ix_j+\lambda x_k^2=0$, $i\neq j\neq k$. One of these, $(k=0)$, is the invariant system of the G_6. Each conic of the G_6 touches at 0 which accounts for six common lines. Among the conics are: two each on six double points, one on six contacts of the double lines which meet at the center, two each on six flexes and one on the six contacts of tangents from 0.

Curve 10. The group is the third types of cyclic G_6 mentioned at the beginning of the section. The multipliers of the homology which generate the cyclic G_3 are 1, 1, ω. The axis of the homology is x_0 and the center is the opposite vertex, while x_1 and the opposite vertex are axis and center of the reflexion. We have here an interesting case since all points on each axis and all lines on the corresponding center must be fixed elements of the associated subgroup. Each side and vertex of the fixed triangle is thus unaltered by the entire G_6. The third side of the invariant triangle, x_2, is a flex-undulation tangent—flex at 0, undulation at ∞—counting for three flexes and seven double lines. The axis of the homology cuts out two triple points and the center carries two triple lines. On the other hand the axis of the reflexion cuts out three double points while the center carries three double lines. We have now accounted for 3 flexes, 9 double points and 16 double lines. The remaining double points form a set of six under the group. Like all general sets they lie in harmonic pairs on three lines through the center of reflexion and in threes on two lines through the center of the homology. The flexes are $2t^{12} + (5a - 14)t^6 + a = 0$ and constitute two general sets.

Curve 11. The G_6 is generated by a homology with multipliers $-\omega^2$, $-\omega^2$, 1. The cyclic subgroup G_3 is likewise generated by a homology with multipliers ω, ω, 1. The two homologies as well as the reflexion have a common center ξ_2 and axis x_2. The center is on the curve at a hyperosculation* point, 0. x_0 is a tangent with 6-point contact (on the axis). The two tangents x_0 and x_1 absorb 9 flexes and 16 double lines. There is an ordinary sixfold point on the axis which accounts for all the double points. Every set of six conjugate points breaks up into harmonic pairs lying on a line through the center. Likewise every set of conjugate lines (tangents) meet in three harmonic pairs in a point on the axis, their contacts forming a set of conjugate points. The flexes, $5t^6 - 7$, thus lie on a line through the center while their tangents meet at a common point on the axis. The contacts of tangents from the point ∞ are a set of conjugate points. Indeed the twelve tangents from an arbitrary point of the axis fall into two sets of conjugate lines whose contacts are two sets of conjugate points. The 24 double lines must divide into four sets of conjugate lines while their contacts make up eight sets of conjugate points.

---

* i. e. a point whose tangent has 7-point contact.

§ 5. *Cyclic Groups of Maximum Order.*

If the order of a cyclic group exceeds 7, the invariant (rational) curve admits an infinite group (Q, § 10). There are three types of septimics with a one-parameter group of transformations—the binomial rational septimics. All the binary transformations have the same two fixed points, 0 and ∞, on each curve. These are parameters of dual singularities at which all the point and line singularities of the curve are concentrated. The fixed triangle of all the ternary transformations is composed of the two singular points and the intersection of their tangents. The simple singularities contained in the compound are enumerated at once by the well-known equivalence formulas.*

Curve 18. There is a hyperosculation point at ∞ whose tangent counts for 5 flexes and 10 double lines. The singularity at 0 is a 6-fold point with coincident parameters, made up of 5 cusps and 10 double points. The 5 cusps absorb the remaining 10 flexes and 30 double lines.

Curve 19. There is a 5-fold point at 0 with coincident parameters and 7-point contact tangent,—equivalent to 4 cusps, 8 double points, 1 flex and 2 double lines. The singularity at ∞ is accordingly made up of 4 flexes, 8 double lines, 1 cusp and 2 double points. As before the 5 cusps account for the other 10 flexes and 30 double lines.

Curve 20. The singularity at ∞ is a triple point with coincident parameters and 7-point contact tangent, accounting for 2 cusps, 4 double points, 3 flexes and 6 double lines. The other singularity is a 4-fold point with coincident parameters and 7-point contact tangent, equivalent to 3 cusps, 6 double points, 2 flexes and 4 double lines. Again the 5 cusps absorb the other 10 flexes and 30 double lines.

Before considering the cyclic groups of order 7, we note some properties of the ρ^n which is invariant under a cyclic G_n.

If a rational curve of order n is invariant under a cyclic group of order n (and no higher group), this group cannot contain a homology.

For the equations of the curve may be taken as

$$\begin{aligned} x_0 &= t^r \\ x_1 &= t^s \qquad r > s > 0, \quad \left\{ \begin{matrix} r \\ s \end{matrix} \right\} < n, \quad r + s > n. \\ x_2 &= t^n + 1, \end{aligned}$$

---

* See e. g. Scott, "On the Higher Singularities of Plane Curves," *American Journal of Mathematics*, Vol. XIV (1892), p. 323, or Hilton, *Plane Algebraic Curves*, p. 119, Ex. 1.

And the first condition evidently excludes equal multipliers since the ternary group is induced by the binary collineation $t' = \epsilon t,\ \epsilon^n = 1$. The first two conditions are equivalent to those given in Q, p. 66; the last is easily obtained. For if $r + s = n$, we have the dihedral G_{2n}. While if $r + s < n$, the transformation $t' = 1/t$ on the parameter which merely effects a renaming of the points, changes r and s into $n - r$ and $n - s$ whose sum will be greater than n. Again

A ρ^n *which is invariant under a cyclic* G_n *(and no higher group) has a multiple point of order greater than 2, having however but two distinct parameters. The parameters of the multiple point are the only fixed points of the binary group and the multiple point is the only fixed point of the ternary group which lies on the curve.*

For it is evident from the parametric equations that the curve has a multiple point with but two parameters of order $n - r + s$, which in consequence of the restrictions on r and s is at least of order 3. The point is a vertex of the reference triangle which is fixed under G_n and it is the only fixed point on the curve since any other would necessarily be a multiple point of order n (S, p. 46).

We take up now the rational septimics admitting cyclic groups of order 7. There are six species, each with a multiple point of order 3 or higher at one vertex of the reference triangle which is the unique fixed triangle of the group. All other points fall into sets of 7 conjugate under G_n. *There can be no other point singularities except ordinary double points.*

Curve 12. The multiple point is a cusp (parameter 0) with 6-point contact tangent, combined with an inflexional point (parameter ∞) with 5-point contact tangent. The flexes are $t^5(5t^7 + 12)$. The cusp branch is of class 4 and absorbs therefore 3 double lines besides 3 flexes. The inflexional branch is likewise of class 4 and uses up three double lines and three flexes. But, to meet the requirements of the group, 8 double points must go to form the multiple point, i. e. the excess is 5. The excess of double lines is then 5 while the cusp absorbs eight more. We have thus accounted for 8 double points and 19 double lines.

Curve 13. The multiple point is a 4-fold point formed by a triple point with 6-point contact tangent (parameter 0) falling at an undulation (parameter ∞). The flexes are $t^6(2t^7 + 9)$. The triple point branch contains two cusps and two flexes. But it is of class 3 and therefore consumes an additional double line. The undulation branch as usual includes two flexes and one double line. But the multiple point must absorb 8 double points, hence the excess is 2. The two cusps reduce the double lines by 15 and we must add two more for excess. There remain 7 double points which must be a general set, and 21 double lines which form three conjugate sets.

Curve 14. The singularity is a 5-fold point composed of a 4-fold point with coincident parameters and 6-point contact tangent (parameter 0), united with a flex (parameter ∞). Since all double points must be concentrated at the multiple point, the excess is 5. The 6-point branch contains three cusps and one flex. The three cusps reduce the number of double lines by 21 to which must be added 5 on account of excess. There remain 14 double lines and 7 flexes: t^7+8.

Curve 15. The singularity is a 6-fold point, caused by a simple branch passing through a 5-fold point with coincident parameters, and counting for 4 cusps and 11 double points. The cusps absorb 26 double lines besides the usual quota of flexes. There remains 7 flexes, t^7+15, and 14 double lines.

Curve 16. The multiple point is formed by a triple point with coincident parameters and 5-point contact tangent (parameter 0) united with a cusp having a 4-point contact tangent (parameter ∞). It is thus a 5-fold point containing three cusps. But it must absorb all the double points since only sets of seven are admissible and the multiple point counts for a minimum of ten. Hence the excess is 5. Each tangent of the multiple point counts once as a flex tangent. 21 double lines go to make up the cusps and 5 more are required to match the nodal excess. There are left then 14 double lines and 7 flexes, the flex form being $t^5(8t^7+15)$.

Curve 17. The singularity is a 6-fold point—formed by the union of an ordinary cusp and a 4-fold point having coincident parameters—equivalent thus to 4 cusps and 11 double points. Since 8 flexes and 26 double lines are required to make the cusps, there are left 7 flexes, $3t^7+10$, and 14 double lines.

The last six curves examined, aside from their intrinsic interest, exemplify the importance of Dr. Scott's theory of excess in the analysis of higher singularities.

Partial List of Ternary Equations.

(5) $x_1^3(x_0+x_1)^4-x_0^3x_2^4=0$

(7) $x_0^5x_2^2-a^4x_1^5x_2^2-x_0^4x_1^3+(1-a)(2a+3)x_0^3x_1x_2^3$
$+2a^2x_0^2x_1^4x_2+a(1-a)^2(a+4)x_0x_1^2x_2^4+(1-a)^5x_2^7=0$

(8) $x_0^5x_2^2 - x_1^2(x_1 + x_2)^5 = 0$

(11) $x_1x_2^6 - x_0(x_0 + x_1)^6 = 0$

(12) $x_0^7 - x_1^7 + 4x_0^4x_1^2x_2 - x_0^2x_1x_2^4 + 2x_0x_1^4x_2^2 = 0$

(13) $x_0^7 + x_1^7 - x_0^3x_1x_2^3 + 3x_0^2x_1^3x_2^2 - 3x_0x_1^5x_2 = 0$

(14) $x_0^7 - x_1^7 - x_0^4x_1x_2^2 + 2x_0^2x_1^4x_2 = 0$

(15) $x_0^7 + x_1^7 - x_0^5x_1x_2 = 0$

(16) $x_0^7 - x_1^7 - 2x_0^5x_1x_2 + x_0^3x_1^2x_2^2$

(17) $x_0^7 + x_1^7 - x_0^4x_1^2x_2 = 0$

(18) $x_1^7 - x_0^6x_2 = 0$, (19) $x_1^7 - x_0^5x_2^2 = 0$, (20) $x_1^7 - x_0^4x_2^3 = 0$

(23) $x_0^7 + x_1^7 - 5x_0^3x_1^3x_2 + 5x_0^2x_1^2x_2^3 - x_0x_1x_2^5 = 0$

(24) $x_0^7 + x_1^7 + 3x_0^3x_1^3x_2 - x_0^2x_1^2x_2^3 = 0$

(25) $x_0^7 + x_1^7 - x_0^3x_1^3x_2 = 0$.

Species of Self-Projective Rational Septimics.

Group	*Generator*	*Parametric Equations.*			
	$t' =$	$x_0 =$	$x_1 =$	$x_2 =$	
C_2	$-t$	$t^7+at^5+bt^3+t$	$t^6+ct^4+dt^2$	t^4+et^2+f	(1)
C_3	ωt	t^7+at^4+bt	ct^6+dt^3+1	t^5+t^2	(2)
C_3	ωt	t^7+at^4	t^4+bt	t^6+ct^3+1	(3)
C_4	it	t^7+at^3	t^5+bt	t^4+1	(4)
C_4	it	t^7	t^3	t^4+1	(5)
C_5	ηt	t^7+at^2	bt^5+1	t^6+t	(6)
C_5	ηt	t^7+at^2	t^5+1	t^4	(7)
C_5	ηt	t^7+t^2	t^5	1	(8)
C_6	$-\omega^2 t$	t^7+at^2	t^6+1	t^2	(9)
C_6	$-\omega^2 t$	t^7+at^2	t^6+1	t^3	(10)
C_6	$-\omega^2 t$	t	t^7	t^6+1	(11)
C_7	ϵt	t^6	t^2	t^7+1	(12)
C_7	ϵt	t^6	t^3	t^7+1	(13)
C_7	ϵt	t^6	t^4	t^7+1	(14)
C_7	ϵt	t^6	t^5	t^7+1	(15)
C_7	ϵt	t^5	t^3	t^7+1	(16)
C_7	ϵt	t^5	t^4	t^7+1	(17)
$C\infty$	at	t^7	t^6	1	(18)
$C\infty$	at	t^7	t^5	1	(19)
$C\infty$	at	t^7	t^4	1	(20)
D_6	ωt	t^7+at^4+bt	bt^6+at^3+1	t^5+t^2	(21)
D_{10}	ηt	t^7+at^2	at^5+1	t^6+t	(22)
D_{14}	ϵt	t^6	t	t^7+1	(23)
D_{14}	ϵt	t^5	t^2	t^7+1	(24)
D_{14}	ϵt	t^4	t^3	t^7+1	(25)

UNIVERSITY OF WASHINGTON,
JUNE, 1923.

Acc. No. Book No. LIBRAR

Third Paper on Tensor Analysis.*

BY G. Y. RAINICH.

This paper deals with some questions of the integral calculus of tensor fields using the method and notions introduced in the foregoing papers.

Before we state, by way of introduction, what questions are treated here, it will be well to point out a question which will not be discussed in the body of the paper. The differentiation of a tensor field of rank k leads to a tensor field of rank $k+1$ (First paper §§ 4 and 9; second paper § 18). It is natural, given a tensor field of rank $k+1$, to ask whether it may be considered as a differential of a field of rank k and, if so, to try to find this tensor field. In the case $k=0$ the given field would be a field of rank *one,* i. e. a vector field, and the questions whether this field is a gradient and, if so, how to find the scalar field of which it is the gradient, are answered in a way entirely analogous to that followed in a Euclidean space. If, however, $k>0$, the situation becomes radically different: let us consider the case $k=1$, for instance, using the customary coordinate notations; if ϕ_i is a tensor field of rank one, if $\phi_{i,j}$ is its first and $\phi_{i,jk}$ its second differential, it is well known that

$$\phi_{i,jk} - \phi_{i,kj} = R^{p}{}_{i;jk}\,\phi_p,$$

where $R^{t}{}_{i;jk}$ is the Riemann tensor of the space. If therefore a tensor field ψ_{ij} of rank two is given we see that, if it is the differential of a tensor field ϕ_i it must satisfy the relation

$$\psi_{ij,k} - \psi_{ik,j} = R^{p}{}_{i;jk}\,\phi_p,$$

which in a general space determines the primitive field completely. We have thus a true "integration by differentiation" and the primitive field is determined uniquely, *there are no constants of integration.* We shall not discuss in this paper these interesting questions, which, it seems, are also important from the physical point of view; we shall not deal here with the research of primitive functions, not with questions which correspond to the study of the indefinite integral but with the process of integration, with the generalization of the definite integral.

We shall follow the system adopted in the first two papers, beginning with the study of the simpler case of a surface in the ordinary space; the indicator, as it ought to be, shows the way; it shows that integration leads away from the surface into the space; we then take up the general case of a tensor field on the surface, show what integration becomes in this case and deduce the integrability conditions or the conditions for the independence of the integral of the path; besides a differential condition which is the exact counterpart of the usual condition we have here additional algebraic conditions. Before entering upon the discussion of the generalization of these things for higher spaces we show how the Riemann tensor can be introduced in a new way which in many respects resembles the introduction of the indicator in the case of a surface. The idea of tensor area seems to prove useful in this connection as well as elsewhere. After we introduce surface integration and volume integration for higher spaces and discuss the integrability conditions following the way adopted in the case of the surface we treat the Riemann tensor and the contracted Riemann tensor as examples. In the last section we indicate a method of obtaining integral expressions without leaving the curved space.*

---

* The two first papers appeared in this *Journal,* Vol. XLVI, No. 2, April, 1924, pp. 71-94 and Vol. XLVII, No. 1, January, 1925, pp. 11-24.

§ 21. *Line Integrals on a Surface.*

We remember that the (negative) indicator was obtained in our first paper (7.5) as

$$\lim \frac{b-a}{\lambda} \quad \text{with} \quad \frac{B-A}{\lambda} \to h,$$

where b and a were unit normal vectors at the points B and A respectively. We may consider the totality of unit normal vectors as a vector field (which, however, does not *belong* to the surface). If we introduce the notation $a=f(A)$, $b=f(B)$ we have

$$-s(h)=\lim \frac{f(B)-f(A)}{\lambda},$$

so that the negative indicator appears as the differential of this vector field (compare 4.1). We may also say that the indicator gives the *infinitesimal* change of the unit normal vector which corresponds to the infinitesimal displacement h on the surface. If we write dA for h, we have

$$da=-s(dA).$$

---

* The contents of this paper was briefly outlined in papers read before the American Mathematical Society on December 28, 1924 and February 27, 1925.

It is natural to consider the *finite* change of the unit normal vector $q - p$ as the integral of $-s(dA)$,

$$q - p = -\int_P^Q s(dA).$$

This suggests going outside of the surface if we want to consider integrals.

Following this suggestion we shall find an operation of integration of which this transition from $s(dA)$ to $q - p$ is a special case. One way to do this is to consider the integral as the limit of a sum. It is, however, not necessary to introduce this process explicitly, as it is possible to reduce the case which we have to consider to things already known. The hint which we have, viz., that we have to take into account the outside space, will help us to make this reduction.

The situation is this. If we have a vector field, i. e. a vector v_A at each point A, we form the integral

$$\int v_A . dA$$

where $v_A . dA$ is the scalar product of the vectors v_A and dA; this can be considered as the limit of the sum

$$v_A . (B - A) + v_B . (C - B) + v_C . (D - C) + \cdots,$$

where $A, B, C, D, \cdots$ are points along a curve, and in coordinates has the familiar form

$$\int v_x dx + v_y dy + v_z dz.$$

Instead of considering a vector v we may consider (property 1, § 3, First Paper) the scalar linear function—tensor of rank one—$\phi(x) = v.x$; our integral takes then the form

$$\int \phi(dA).$$

If we have, say, a tensor of rank two, $\phi(x, y)$ and our surface happens to be *a plane*, we may give to the second argument y a fixed value and form, without any difficulty, the integral

$$\int \phi(dA, y),$$

(for every given path of integration) a function of y, to be precise a tensor of rank one whose argument is y, instead of a number which we obtained before—and the same can be applied to tensors of higher ranks. But, if we have a curved surface, we encounter the same difficulty as in the case of differentiation (§ 9): we cannot give to y a fixed value. We therefore resort to the *following procedure* which is of fundamental importance for all that follows:

We take a fixed, but arbitrary vector c of the space which contains our surface; we consider in every point A of the surface a vector c_A which is the projection of c on that bundle (on the tangent plane at A) and in forming the integral, we give to y at each point A the value c_A, viz.,

$$\int \phi(dA, c_A).$$

We obtain thus (for every path of integration) a number, say ψ. Now we make c variable (this corresponds to the restoration of variability to y in the case of the plane). The number ψ becomes a function of the vector c, and, as we easily see a linear function (using the fact that the projection of the sum of two vectors is the sum of their projections and the definition of a linear function in § 3, First paper). We thus obtain as a result of integration of a tensor field $\phi(x, y)$ of rank two (for a given path) a tensor of rank *one* but this tensor is *a tensor of the three dimensional containing space.* We may, again, consider instead of the tensor $\psi(c)$ the vector u defined by the relation $\psi(c) = u . c$ (§ 3) and instead of the tensor of the second rank $\phi(x, y)$ we may consider, at each point of the surface, the equivalent linear vector function $g(x)$ which is given by $g(x) . y = \phi(x, y)$ and say that the result of integration of a linear vector function, which belongs to a surface, along a path on this surface is a vector of the containing space.

The indicator being a linear vector function, and the difference between the unit normal vectors at the ends of a path being a vector of the containing space, we see easily that we have actually found the generalization which we set out to find.

The further generalization for fields of higher ranks is obvious. In order not to complicate the writing we shall only mention the case of rank three; we take in this case *two* arbitrary vectors of the containing space c and q, consider their projections c_A and q_A, form as before the tensor field of rank one $\phi(dA, c_A, q_A)$, integrate this along a line Γ and obtain a tensor of the containing space $\psi\ (c, q)$ which has to be considered as the integral:

$$\int_\Gamma \phi(dA, c_A, q_A) = \psi(c, q).$$

If the given tensor field is not symmetric we may also form two other integrals

$$\int_\Gamma \phi(c, dA, q) \quad \text{and} \quad \int_\Gamma \phi(c, q, dA).$$

§ 22. *Integrability conditions.*

In the case of the indicator we know, from the geometrical meaning of the integral $\int\limits_{\Gamma} s(dA)$, which gives the change of the unit normal vector along the curve Γ, that if two curves Γ_1 and Γ_2 have the same end points

$$\int\limits_{\Gamma_1} s(dA) = \int\limits_{\Gamma_2} s(dA),$$

provided the curves may be continuously transformed one into the other without leaving the surface, because each of the two integrals represents the difference between the normal unit vectors at the ends of the path. This suggests considering the general question of when, given an integral of a tensor field of rank two, it depends only on the end points of the path of integration and not on the path itself or when the integral taken over a closed reducible contour vanishes. In order that this be so $\int\limits_{\Gamma} \phi(da, c_A)$ must be independent of the path for every value of c. If we put, for a given c, $\phi(x, c_A) = \omega(x)$ we know that the condition of independence is

22.1 $$\omega'(x, h) = \omega'(h, x)$$

where $\omega'(x, h)$ is the differential of the tensor field $\omega(x)$. (It would lead us too far if we desired to give here the vector deduction of this result which is well known in coordinate form). We have, therefore, to find the differential of the tensor field $\phi(x_A, c_A) = \omega(x_A)$, where c_A is the projection of a *constant* vector of the containing space. We know (First paper, § 9) that the differential of $\omega(x_A)$ or, to be more explicit, $\omega(A, x_A)$ is given by

$$\omega'(A, x_A, h_A) = \lim \frac{\omega(B, x_{A.B}) - \omega(A, x_A)}{\lambda},$$

$$\text{with } \frac{B - A}{\lambda} \rightarrow h_A$$

where $x_{A.B}$ is the projection of x_A on the bundle B. We therefore have

22.2 $$\omega'(A, x_A, h_A) = \lim \frac{\phi(B, x_{A.B}, c_B) - \phi(A, x_A, c_A)}{\lambda}.$$

This is *not* the differential of the tensor field of the second rank $\phi(A, x_A, y_A)$ for $y_A = c_A$. To have the differential we must have in the first term of the numerator $c_{A.B}$, the projection of c_A on the bundle B, instead of c_B, the projection of c on the bundle B. However, this coincides with the differential $\phi'(A, x_A, c_A)$, if c coincides with c_A, i. e. if the vector c belongs (or is parallel) to the bundle A. In this case 22.1 takes the form

22.3 $$\phi'(x_A, c_A, h_A) = \phi'(h_A, c_A, x_A).$$

Since 22.1 must hold for all values of c, 22.3 must be satisfied, it is a *necessary* condition of integrability; in order to find sufficient conditions we will have to consider the case when c *does not* belong to the bundle A. Before we do this we notice that, applied to the indicator, equation 22.3 gives us the Codazzi equations of which we have thus found a derivation which seems to be novel.

Coming now to the general case we may present c as the sum of a vector y_A of the bundle A and of a vector which is normal to this bundle and which we may write ρa, a being the unit normal vector and ρ a number. We have thus

$$c = y_A + \rho a, \quad c_A = y_A, \quad c_B = y_{A.B} + \rho a_B;$$

22.2 takes the form

22.21 $$\omega'(A, x_A, h_A) = \lim \frac{\phi(B, x_{A.B}, y_{A.B}) - \phi(A, x_A, y_A) + \rho\phi(A, x_A, a_B)}{\lambda}$$
$$= \phi'(A, x_A, y_A, h_A) + \rho\phi(A, x_A, \lim a_B/\lambda)$$

We have to find now $\lim a_B/\lambda$. The vector a_B is the projection of the unit normal vector at A on the tangent plane at B and is easily shown to be $a - b(ab)$ using the reasoning employed in proving formula 9.2. We have now

$$\lim a_B/\lambda = \lim \frac{a - b(ab)}{\lambda} = \lim \frac{a - b}{\lambda} + \lim \frac{b[(a-b).a]}{\lambda} = s(h)$$

(compare the similar computation in § 12). We have thus

22.4 $$\omega'(A, x_A, h_A) = \phi'(A, x_A, y_A, h_A) + \rho\phi(A, x_A, s(h_A)).$$

Since 22.1 must hold for an arbitrary c we have to consider here y_A and ρ as arbitrary. Making $\rho = 0$ we obtain once more condition 22.3; making $y_A = 0$, $\rho = 1$ we obtain from 22.1

22.5 $$\phi(x, s(h)) = \phi(h, s(x))$$

where the letter A is dropped as argument and as index. This together with 22.3 which we write more conveniently as

$$\phi'(x, y, h) = \phi'(h, y, x), \tag{22.6}$$

gives a set of necessary and sufficient conditions of integrability.

If we have the integrand in the form of a vector function $f(x)$, the condition 22.5 becomes

$$f(x) \,.\, s(h) = f(h) \,.\, s(x) \tag{22.7}$$

which may also be written as

$$sf(x) \,.\, h = sf(h) \,.\, x \tag{22.71}$$

and, in the case when f is symmetric, in the even simpler form

CALCUTTA UNIVERSITY LIBRARY
Acc. No.........
Book No.........

$$sf(x) = fs(x) \quad \text{or} \quad sf = fs \tag{22.72}$$

which shows that the function f has to be commutative with the indicator s; condition 22.6 takes here the form

$$f'(x, h) = f'(h, x). \tag{22.8}$$

In the case of the indicator condition 22.7 is clearly satisfied, both sides becoming identically equal.

It may not be superfluous to insist on the fact that we have here, besides the usual differential condition of integrability, an additional algebraic condition involving the indicator. If both these conditions are satisfied for a tensor field of rank two which belongs to the surface there exists an ∞^1 of vector fields which are defined at all points of the surface but do not *belong* to the surface, because their vectors instead of being tangent stick out of it. All these fields may be obtained from one of them by the addition to all vectors of an arbitrary constant (space) vector.

If the integrand is a field of rank three $\phi(x, y, z)$ we have to put $c = y_A + \rho a$, $q = z_A + \sigma a$; we obtain instead of 22.21 and 22.4

$$\omega'(A, x_A, h_A) = \lim \frac{\phi(B, x_{A.B}, y_{A.B}, z_{A.B}) - \phi(A, x_A, y_A, z_A)}{\lambda} +$$

$$+ \lim \frac{\rho\phi(A, x_A, a_B, z_A) - \sigma\phi(A, x_A, y_A, a_B) + \rho\sigma\phi(A, x_A, a_B, a_B)}{\lambda}$$

$$= \phi'(A, x_A, y_A, z_A, h_A) + \rho\phi(A, x_A, s(h_A), z_A) + \sigma\phi(A, x_A, y_A, s(h_A))$$

the term containing a_B twice vanishing in the limit.

The integrability condition consists of three relations, one differential:

$$\phi'(x, y, z, h) = \phi'(h, y, z, x) \tag{22.61}$$

and two algebraic:

22.51 $$\phi(x, s(h), z) = \phi(h, s(x), z)$$

and

22.52 $$\phi(x, y, s(h)) = \phi(h, y, s(x))$$

§ 23. *Line Integrals in Higher Spaces.*

If, instead of a surface of the ordinary space, we consider an n-S. immersed in an N.-E.S., line integrals of tensor fields can be introduced in exactly the same way. In deducing the integrability condition a change has to be made, however. The vector c which belongs to the containing space must be presented as $y_A + a$, where a is an arbitrary normal vector; we cannot put it in the form ρa there being no definite unit normal vector when $r > 1$. Formula 22.21 takes the form

23.1 $$\omega'(A, x_A, h_A) = \phi'(A, x_A, y_A, h_A) + \phi(A, x_A, \lim a_B/\lambda)$$

where a is an arbitrary normal vector. We have now to find $\lim a_B/\lambda$ in this case.

We shall use this occasion and take up again the question of introducing the indicator dealt with in § 17 of the second paper and then in the added remark at the end of it. We consider at each point B of the neighborhood of A, B approaching A in such a way that $(B - A)/\lambda \rightarrow k_A$, a tangential vector h_B and a normal vector b, the values of these vectors for A being h_A and a respectively. We have then

23.2 $$h_B . b = 0 \qquad h_A . a = 0$$

We suppose that $\dfrac{h_B - h_A}{\lambda}$ and $\dfrac{b - a}{\lambda}$ approach finite limits as λ approaches zero so that

$$\lim \frac{(h_B - h_A)(b - a)}{\lambda} = 0.$$

This gives

$$\lim \frac{h_B . b}{\lambda} - \lim \frac{h_A . b}{\lambda} = \lim \frac{h_B . a}{\lambda} - \lim \frac{h_A . a}{\lambda}$$

or in view of 23.2

23.3 $$-\lim \frac{h_A . b}{\lambda} = \lim \frac{h_B . a}{\lambda} .$$

Since the left hand side of the equality does not contain h_B and the right hand side does not contain b we conclude that the common value of both sides does not depend on either. This common value is, therefore, only a function of the vectors h_A, a and, of course, of k_A. Let us denote this function which, obviously, is a linear function of each of the three vectors, by $\psi(h_A, k_A, a)$, so that we have

$$\psi(h_A, k_A, a) = -\lim \frac{h_A . b}{\lambda} = \lim \frac{h_B . a}{\lambda} . \tag{23.4}$$

Using property 3 of § 3 (First paper) we may write ψ also in one of the following forms

$$\psi(h_A, k_A, a) = h_A \mathfrak{J}(a, k_A) = a . s(h_A, k_A) \tag{23.5}$$

where $\mathfrak{J}(a, k_A)$ is the tangential component of $\lim \frac{a - b}{\lambda}$ and $s(h_A, k_A)$ the normal component of $\frac{h_B - h_A}{\lambda}$. The functions $\mathfrak{J}$ and s have been introduced in the second paper ($\mathfrak{J}$ with the opposite sign). Here $\mathfrak{J}$ (a, k_A) is a linear vector function one of whose arguments is a tangential and the other a normal vector, while its value is a tangential vector; both arguments of s are tangential vectors and its value is normal; ψ is a scalar trilinear function with two tangential and one normal argument. The functions ψ, $\mathfrak{J}$, s are only different forms of the same thing. We shall refer to all three of them indifferently as to the indicatrix (reserving the name indicator for the function $s(x)$ which exists only in the case $N - n = 1$).

We return now to our question: What is $\lim a_B/\lambda$? It is clear, first of all, that it is a tangential vector. Denoting by a_b the normal component of a at the point B we may write

$$a_B = a - a_b$$

and

$$\lim \frac{a_B . h_A}{\lambda} = \lim \frac{(a - a_b) . h_A}{\lambda} = -\lim \frac{a_b . h_A}{\lambda} .$$

Making now $b = a_b$ in the formulas 23.4 and 23.5 we may write

$$h_A . \mathfrak{J}(a, k_A) = \lim \frac{a_b . h_A}{\lambda} = -\lim \frac{a_B h_A}{\lambda} = -h_A . \lim \frac{a_B}{\lambda}$$

and, since $\lim a_B/\lambda$ is tangential, as remarked,

$$\lim a_B/\lambda = - \mathfrak{J}(a, k_A). \tag{23.6}$$

There would be some advantage in taking this as the definition of $\mathfrak{J}$. Relation 23.1 becomes now

$$\omega'(x, h) = \phi'(x, y, h) - \phi(x, \mathfrak{J}(a, h)).$$

Since 22.1 must hold for all values of $c = y + a$ it must hold for $a = 0$ and $y = 0$ so that we have

$$\phi'(x, y, h) = \phi'(h, y, x)$$

and

23.7 $$\phi(x, \mathfrak{F}(a, h)) = \phi(h, \mathfrak{F}(a, x))$$

and this must hold for every normal vector a. The formula 23.7 gives us the required algebraic integrability condition in the case now under consideration; relation 22.5 is, of course, a special case of this, $\mathfrak{F}(a, h)$ becoming $s(h)$ in the case when there exists a unique unit normal vector.

Using the linear vector function $f(x)$ instead of the tensor of the second rank ϕ we may write our relation as

$$f(x) . \mathfrak{F}(h, a) = f(h) . \mathfrak{F}(x, a)$$

or, with the aid of 23.5,

$$s[f(x), h] . a = s[f(h) . x] . a.$$

Since this must hold for an arbitrary a,

$$s[f(x), h] = s[f(h), x].$$

The generalization of 23.7 for the case of a tensor of higher rank is immediate. For rank three we have, for instance, outside of the differential condition the two algebraic conditions

23.8 $$\phi[x, \mathfrak{F}(a, h), z] = \phi[h, \mathfrak{F}(a, x), z]$$

and

23.81 $$\phi[x, y, \mathfrak{F}(a, h)] = \phi[h, y, \mathfrak{F}(a, x)].$$

§ 24. *The Tensor Area.*

In higher spaces the Riemann tensor plays in some respects the same rôle that the indicator plays in the case of an ordinary surface (and, more generally, in the case $N - n = 1$). We considered the indicator as giving the infinitesimal change of the unit normal vector which corresponds to an infinitesimal displacement along a curve on the surface, and its integral gave us the finite change of the unit normal vector which corresponds to a finite displacement along a curve. In the case which we are about to consider now the rôle played by the curve will be played by a surface belonging to the n-S. considered; instead of the change of the normal unit vector between the two ends of a finite curve (which constitute the frontier of that curve) we shall consider the change in the orientation of normal spaces which corresponds to the frontier of a finite piece of surface, i. e. to a contour, and we shall see that in the case of an infinitesimal contour the corresponding infinitesimal change of the normal spaces is connected with the Riemann tensor. Before we come to this, however, we have to introduce a new notion which has the same relation to the frontier of a piece of surface as the vector chord, i. e. the vector joining the ends of an arc, has to these two points which constitute the frontier of this arc. We consider here an N.-E.S.

The vector which joins the points A and B we agreed to represent by $B - A$; we know also that the consideration of a vector is equivalent to the consideration of its product with a variable vector—a linear scalar function (First paper, 3.4). In our case this tensor is $(B-A).x$, but it is more convenient to write it in the more complicated form

$$a(x) = \begin{vmatrix} 1 & (A-P).x \\ 1 & (B-P).x \end{vmatrix},$$

where P is an arbitrary point on which the expression obviously does not depend.

It is suggested, if we are given three points A, B, C, to consider the expression

$$\text{24.1} \qquad \beta(A,B,C;\, x,y) = \tfrac{1}{2} \begin{vmatrix} 1 & (A-P).x & (A-P).y \\ 1 & (B-P).x & (B-P).y \\ 1 & (C-P).x & (C-P).y \end{vmatrix}.$$

This expression also does not depend on P; it is for given A, B, C a tensor of the second rank which is alternating or antisymmetric in the variables x, y, i. e.

$$\beta(x,y) = -\beta(y,x).$$

In the case of a plane, $\beta(x,y)$ is simply the area of the triangle ABC multiplied by the so-called alternating product of the vectors x and y; in the case when $N = 3$, i. e. in ordinary space, $\beta(A,B,C;x,y)$ can be presented in the form of the scalar product of a certain vector v by the vector product of x and y; this vector v is perpendicular to the triangle ABC and its length is equal to the area of this triangle; it is called the vector area of the triangle or a bivector; the vector area of a triangle as well as its generalization for any contour have been considered by Peano, Laisant and Fréchet (see the latter's paper "Sur une généralisation des notions d'aire et de plan," *Nouv. An.* (4), 4 (1904), pp. 241-248); the present writer used it in connection with integration in space (see § 27 below).

When $N > 3$ it is impossible to get rid of the variable vectors x and y as in the cases $N = 2$ and $N = 3$ so that we have to consider the thing as a function of x and y. Schouten uses the term bivector also in this case. (Compare e. g. J. A. Schouten and D. J. Struik, "Einführung in die neueren Methoden der Differentialgeometrie," I, § 1.) It should be noticed that making $P = A$ we obtain a simpler but less symmetrical expression for β, viz.

24.11 $$\beta(A, B, C; x, y) = \tfrac{1}{2} \begin{vmatrix} (B-A).x & (B-A).y \\ (C-A).x & (C-A).y \end{vmatrix}.$$

If we are given a rectilinear contour $A, B, C, D, \cdots, K, L$ or a number of points in a given cyclic order, we form

24.2 $$\begin{aligned}\beta(A, B, C, \cdots, K, L, A; x, y) &= \beta(A, B, P; x, y) \\ &+ \beta(B, C, P; x, y) + \beta(C, D, P;\ x, y) + \cdots \\ &+ \beta(K, L, P;\ x, y) + \beta(L, A, P;\ x, y),\end{aligned}$$

an expression which, obviously, does not depend on P; we call this the tensor area of the contour $ABCD \cdots KL$. Given now a curvilinear contour Γ, we may approximate it by rectilinear contours and define the tensor area of Γ, which we shall denote by $\beta(\Gamma; x, y)$, as the limit of the tensor areas of these approximation contours when this limit exists, which happens in a very general case.

The notion may be extended by starting with four, or any number of points, instead of three; we have for the "tensor volume" of a tetrahedron $ABCD$ the expression

24.3 $$\gamma(A, B, C, D; x, y, z) = \tfrac{1}{6} \begin{vmatrix} 1 & (A-P).x & (A-P).y & (A-P).z \\ 1 & (B-P).x & (B-P).y & (B-P).z \\ 1 & (C-P).x & (C-P).y & (C-P).z \\ 1 & (D-P).x & (D-P).y & (D-P).z \end{vmatrix}.$$

If we have any number of points (> 4) in a given spherical order (as analogous to a cyclic order) we may form the tensor volume for such a polyhedron and by a limit process extend the notion to apply to a closed surface etc.

§ 25. *Contour variation of normals and a new introduction of the Riemann tensor.*

We introduced the indicator with the aid of the change $b - a$ of the normal unit vector; instead of the vector $b - a$ we may here again write the tensor of the first rank $(b - a).x$ or

25.1 $$\begin{vmatrix} 1 & a.x \\ 1 & b.x \end{vmatrix}.$$

If $N - n = r > 1$ this cannot be generalized immediately. We denote as before the tangent bundle at a point A by the same capital letter A, and the normal bundle we denote by the corresponding small letter a. The projection of a vector x on the normal bundle we denote by affixing this letter a: thus x_a; in the case $r = 1$ this would be $a(a.x)$. The expression 25.1 suggests the formation of

25.2 $$\delta(A, B, C; x, y) = \begin{vmatrix} 1 & x_a & y_a \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{vmatrix}$$

when we are given three points A, B and C; this is a tensor of the N.-E.S. which characterizes the relative positions of the normal r-dimensional Euclidean spaces to one curved n-S. at the three points A, B, and C. It is, like the tensor area, an alternating tensor and, exactly as in the case of the tensor area, the definition of this tensor, which we call *the variation of normals* (normal Euclidean spaces or normal bundles) may be extended to apply to any rectilinear, and then to a curvilinear contour by first forming the expression

$$\delta(A, B, P; x, y) + \delta(B, C, P; x, y) + \cdots + \delta(L, A, P; x, y)$$

and then passing to the limit.

We thus obtain for a contour Γ a tensor $\delta(\Gamma; x, y)$ *the variation of normals* to the n-S. *for the contour* Γ. This corresponds to the finite change of the unit normal vector between two points.

Now, according to our program, we pass to the infinitesimal variation, which corresponds to an infinitesimal contour. We begin with a triangular contour A, B, C considering A as a fixed point and B and C as variable points approaching A in such a way that $\dfrac{B - A}{\lambda} \rightarrow h_A$, $\dfrac{C - A}{\mu} \rightarrow k_A$. We write 25.2 in the form

25.21 $$\delta(A, B, C; x, y) = \begin{vmatrix} x_b - x_a & y_b - y_a \\ x_c - x_a & y_c - y_a \end{vmatrix}.$$

If x and y are vectors of the tangent bundle at A we have $x_a = 0$, $y_a = 0$;

we may write in this case x_A for x and y_A for y; x_b will be $x_A - x_{A.B}$ etc., so that

$$\delta(A, B, C; x_A, y_A) = \begin{vmatrix} x_{A.B} - x_A & y_{A.B} - y_A \\ x_{A.C} - x_A & y_{A.C} - y_A \end{vmatrix}.$$

Dividing both parts by $\lambda\mu$ and passing to the limit we have (§ 17, Second paper) on the right hand side the Riemann tensor at the point A. A connection is thus established between the infinitesimal variation of normals and the Riemann tensor.

But this is not the whole story; the Riemann tensor gives us only *those* values of the tensor expressing the infinitesimal variation which correspond to the values of x and y *belonging to the curved space.* In order to know the infinitesimal variation of the normals completely we have to consider not only $\delta(x_A, y_A)$ but $\delta(x, y)$ for *all* values x, y. If we write in the general case $x = x_A + x_a$, $y = y_A + y_a$, we have

$$x_b - x_a = x_A - x_{A.B} + x_{ab} - x_a,$$

where x_{ab} denotes the projection of the normal component (with respect to the point A) x_a of x on the normal bundle at B; but then $x_{ab} - x_a = -(x_u)_B$ and we have (23.6)

$$\lim \frac{(x_a)_B}{\lambda} = -\mathfrak{F}(x_a, h)$$

The full expression for the infinitesimal variation is therefore

$$\begin{vmatrix} s(x_A, h_A) + \mathfrak{F}(x_a, h_A) & s(x_A, k_A) + \mathfrak{F}(x_a, k_A) \\ s(y_A, h_A) + \mathfrak{F}(y_a, h_A) & s(y_A, k_A) + \mathfrak{F}(y_a, k_A) \end{vmatrix}.$$

Since s and $\mathfrak{F}$ are perpendicular to each other, s being a normal vector and $\mathfrak{F}$ a tangential one (see 23.5), the products of the type $s.\mathfrak{F}$ vanish and we only have

$$25.4 \qquad \begin{vmatrix} s(x_A, h_A) & s(x_A, k_A) \\ s(y_A, h_A) & s(y_A, k_A) \end{vmatrix} + \begin{vmatrix} \mathfrak{F}(x_a, h_A) & \mathfrak{F}(x_a, k_A) \\ \mathfrak{F}(y_a, h_A) & \mathfrak{F}(y_a, k_A) \end{vmatrix}.$$

The first term is the Riemann tensor. We see thus that the Riemann tensor gives us only *a part* of the infinitestimal variation of normals; we may say that it gives all the information concerning the variation which can be expressed in terms of the curved space, of its *internal* properties, because the other part involves vectors x_a, y_a which require the consideration of the containing space, i. e. which refer to external properties (compare § 17, Second paper).

Here we have a radical difference between the Riemann tensor in general space and the indicator of a surface: the latter gives us full information concerning the change of the unit normal vector because this change can be controlled from the surface; the variation of normals cannot be controlled from the space (in general) and the Riemann tensor, which is an internal thing, only gives us partial information.

But there are cases where the *whole* variation can be given by the Riemann tensor; this happens obviously, when the second part of 25.4 vanishes, i. e. for spaces for which we have

$$\mathcal{J}(x_a, h) \, . \, \mathcal{J}(y_a, k) = \mathcal{J}(x_a, k) \, . \, \mathcal{J}(y_a, h) \, ; \tag{25.5}$$

in these cases, we may say, the whole infinitesimal variation lies in the curved space and is, therefore, given by the Riemann tensor. Only in these cases can we hope to obtain the finite variation of normals, which corresponds to a finite contour, by integrating the Riemann tensor in analogy to what we had in the case of a surface.

We shall not investigate here the peculiarities of spaces for which condition 25.5 is satisfied; this would lead us to considerations which lie outside the scope of the present paper; in dealing with surface integrals we will, however, again be led to relation 25.5.

We want now to get rid of the restriction which we imposed upon ourselves by considering only *triangular* infinitesimal contours. Let us consider a rectilinear infinitesimal contour, constituted by the points $A+h$, $A+k$, $A+l$, $A+m, \cdots, A+n$, $A+h$. It is clear from the preceding that the corresponding variation of normals is given by $\rho(x, y; h, k) + \rho(x, y; k, l) + \rho(x, y; l, m) + \cdots + \rho(x, y; n, h)$.

From the expression 17.2 for the Riemann tensor we conclude that, for fixed x and y, $\rho(x, y; h, k)$ is a sum of terms of the type

$$\begin{vmatrix} ph & pk \\ qh & qk \end{vmatrix}$$

where p and q are vectors belonging to the bundle A. The infinitesimal variation of normals which we are considering has, therefore, for its expression

$$\Sigma \left\{ \begin{vmatrix} ph & pk \\ qh & qk \end{vmatrix} + \begin{vmatrix} pk & pl \\ qk & ql \end{vmatrix} + \begin{vmatrix} pl & pm \\ ql & qm \end{vmatrix} + \cdots \begin{vmatrix} pn & ph \\ qn & qh \end{vmatrix} \right\} \tag{25.6}$$

where the summation is extended over a certain number of pairs of vector p_1, q_1, p_2, q_2 etc.

Now the tensor area of the contour $A+h, A+k, \cdots, A+n$ is

25.7 $$\begin{vmatrix} xh & xk \\ yh & yk \end{vmatrix} + \begin{vmatrix} xk & xl \\ yk & yl \end{vmatrix} + \begin{vmatrix} xl & xm \\ yl & ym \end{vmatrix} + \cdots \begin{vmatrix} xn & xh \\ yn & yh \end{vmatrix}.$$

Suppose that we take another contour *with the same tensor area;* this means that 25.7 has for the new values of $h, k, l, \cdots, m, n$ and for *arbitrary* values of x, y, the same value as for the original $h, k, l, \cdots, m, n$. Making x, y in turn equal to p_1, q_1; p_2, q_2 etc. we see that all the terms of 25.6 do not change when we substitute new values of $h, k, l, \cdots, m, n$ for the old ones, provided the new contour has the same tensor area. *The infinitesimal change of normals for an infinitesimal contour* Γ *in the neighborhood of a point A of the n-S. is thus seen to depend solely on the tensor area of this contour;* and of course it is a linear function of this tensor area. This completes the analogy with the indicator considered as a linear function of the displacement vector giving the infinitesimal change of the unit normal vector.

We may add that this gives a plausible interpretation of the Riemann tensor as a *bivector tensor* as Schouten puts it (loco citato, p. 34).

§ 26. *Contraction.*

In order to deal with integrals extended over surfaces, volumes etc. we must now introduce the operation of contraction, the use of which we have been able to avoid hitherto.

We start with the identity

26.1 $$x = \sum_{\nu=1}^{n} i_\nu \, (i_\nu x)$$

where $i_1, i_2, \cdots, i_n$ are n mutually perpendicular unit vectors of an n-dimensional bundle and which holds for every vector x of the bundle. We shall have occasion to use expressions of this type, i. e. sums of n terms each term of which contains twice one of the n mutually perpendicular unit vectors; if we have a tensor or a multilinear vector function $\phi(x, y, z, \cdots)$, the expression which we have in mind is

$$\sum_{\nu=1}^{n} \phi(i_\nu, i_\nu, z, \cdots).$$

It is easy to show that such an expression does not depend upon the particular set of n mutually perpendicular unit vectors which we use, i. e. that it does not change if we substitute one such set for another. We therefore introduce for such a sum the notation $\phi(I, I, z \cdot \cdot \cdot)$ replacing each of the two variable vectors for which i_ν stand in the sum by a Roman numeral. If such an operation has to be performed more than once on the same tensor we use each time another numeral, so that $\phi(I, II, I, II)$ stands for

$$\sum_{\nu=1}^{n} \phi(I, i_\nu, I, i_\nu) .$$

This operation is essentially the same as the operation of contraction (Verjüngung) of tensors in the usual form of tensor calculus with the difference that we have here no distinction between lower and upper indices. The Roman numerals stand here for the dummy suffices.

With the notation introduced, identity 26.1 takes the form

26.11 $$x = I(Ix)$$

multiplying this by y we obtain

26.2 $$xy = (Ix)(Iy)$$

which means that the product of two vectors is the sum of the products of their components.

If we have a scalar linear function $\phi(x)$—a tensor of rank one—we may write

26.3 $$\phi(x) = \phi[I(Ix)] = \phi(I)(Ix) = [\phi(I) . I] . x \ .$$

The expression $\phi(I)I$ is a vector which written explicitly is

$$\phi(I)I = \sum_{\nu=1}^{n} \phi(i_\nu)\, i_\nu$$

if we denote this vector by g, we have $\phi(x) = g . x$. The operation of contraction permits us thus to write explicitly in terms of a scalar linear function the vector which is the constant factor of it (compare property 1 § 3, First paper).

It may be said that our definition of contraction contains traces of a coordinate system; this may be avoided to a certain extent by using the vector product of $n - 1$ vectors but the expression, though more intrinsic, becomes more cumbersome and it was not thought worth while to introduce it here. We only shall mention that in the case of 3 dimensions we have

$$\phi(I, I) . (\overline{xyz}) = \phi(x, y\,{}^\wedge z) + \phi(y, z\,{}^\wedge x) + \phi(z, x\,{}^\wedge y)$$

where $x\,{}^\wedge y$ denotes the vector product and $\overline{xyz}$ the triple scalar product, and

x, y, z are any three not co-planar vectors and in the case of four dimensions we have similarly,

$$\phi(I,I)\,.\,\overline{(xyzu)} = \phi(\overline{xyz}, u) - \phi(\overline{yzu}, x) + \phi(\overline{zux}, y) - \phi(\overline{uxy}, z)$$

where $\overline{xyz}$ is the vector product and $\overline{xyzu}$ the quadruple scalar product.

The operation of contraction may be used to pass from one form of the indicatrix to another; for instance (23.5)

$$\mathcal{F}(a, x) = \psi(x, I, a)\,.\,I = [a\,.\,s(x, I)]\,.\,I.$$

We are not going to enter into the details concerning the products mentioned above; the vector algebra in 4 as well as in 3 dimensions can be developed intrinsically (see e. g. a paper by A. Mochoolsky in the Memoirs of the Research Institute, Odessa, February, 1924 (in Russian)); here for the sake of completeness we only shall state that, in three dimensions the vector product $x\hat{}y$ is the vector whose components are the determinants of the matrix

$$\left\Vert \begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \right\Vert$$

the triple scalar product $\overline{xyz}$ in three dimensions is the three row determinant formed with the components of x, y, z; the vector product $\overline{xyz}$ in four dimensions is the vector whose components are determinants of the matrix

$$\left\Vert \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right\Vert$$

and the scalar quadruple product is the four-row determinant of the components of four vectors.

§ 27. *Surface Integrals in Curved Space.*

If we have in a n-S. an alternating (antisymmetric) tensor field of the second rank $\phi(A, x_A, y_A)$ and a surface Σ, we may integrate this field over the surface. If we imagine the surface divided into "infinitesimal triangles" so that two of the sides of each triangle may be considered as displacements of one of its vertices A, we form the sum of expressions $\phi(A, dA, \delta A)$ and this sum might be considered as the surface integral. But it is more satisfactory to consider any finite division of the surface by means of a net of curves on it. We form the tensor area $\beta(\Gamma_\nu;\, x, y)$ for the contour which constitutes the frontier of each part, we multiply this by the value $\phi(A_\nu, x, y)$ of the given tensor field at some point A_ν belonging to the same part of the surface and contract this product so as to obtain

$$\phi(A_\nu, I, II) : \beta(\Gamma_\nu; I, II);$$

we form the sum of these expressions for all the parts into which the surface was divided and consider the integral as the limit (if any) to which these sums tend when the diameters of the parts (i. e. the upper limits of the distances between two points belonging to the same part) tend toward zero.

It is not the purpose of this paper to enter into details concerning the definition of integrals and the conditions under which they exist; we are interested here in questions of a less subtle nature; it might be interesting, however, to notice that the above definition of a surface integral *does not suppose the existence of tangent planes to the surface.* Integration in ordinary space was considered from a similar point of view in a paper by the present writer which appeared (in Russian) in the *Bulletin of the Physico-Mathematical Society of Kazan* in 1915.

If we consider the surface in a *Euclidean* space and have in this space a tensor field of a rank higher than two, which is alternating in two arguments, say $\phi(A, x, y, z, \cdots)$, with

$$\phi(A, x, y, z, \cdots) = -\phi(A, y, x, z, \cdots),$$

we may integrate it over the surface by giving to the "superfluous" arguments z etc. fixed values. But if we consider the surface in a *curved n*-dimensional space this is not possible and we have to resort to the device, used in § 21, of giving to z and other arguments values which are projections of fixed vectors in the containing N.-E.S. We obtain thus as the result of integration of a tensor field of rank k (which is alternating in two of its arguments) over a surface a tensor of the containing space of rank $k-2$.

Integration over a volume, i. e. a spread of 3 dimensions presents after this no difficulties: the integrand must be a tensor field alternating in three of its arguments; we divide the volume in parts; combine the tensor volume of the frontier of each part with the value of the integrand at some point of this part, by multiplying and then contracting three times; we give to superfluous arguments values which are projections of fixed vectors of the containing space; form the sum of the numbers so obtained; and pass to the limit making the diameters of the parts tend toward zero. The integral is a tensor of rank $k-3$ of the containing space.

§ 28. *Integrability Conditions for Surface Integrals.*

We shall take over from the usual coordinate exposition, similarly to what was done in § 22, the integrability condition for the surface integral of a tensor field of rank two. This condition, viz., the condition that an integral of $\phi(A, x_A, y_A)$ over a surface depends only on the contour of the surface, i. e. that it is the same for two surfaces having for their frontiers the same contour provided that one can be continuously transformed into the other without leaving the part of the n-S. in which $\phi(A, x_A, y_A)$ is considered, is that in this part ϕ satisfies the relation

28.1 $$\phi'(x, y, h) + \phi'(y, h, x) + \phi'(h, x, y) = 0,$$

where $\phi'(x, y, h)$ is the differential of $\phi(x, y)$.

If we are given a tensor field of higher rank than two, let us say, for example, a tensor field of rank four $\phi(x, y, z, u)$ which is alternating in the arguments x, y, and we integrate it with respect to these arguments using the method indicated in § 27 we have to write that the integrability conditions are satisfied for any tensor field $\omega(x, y)$ which is obtained from $\phi(x, y, z, u)$ with the aid of two vectors c, q of the N.-E.S. which contains the n-S. in which the tensor field ϕ and the surfaces over which we integrate are given, viz.,

$$\omega(A, x_A, y_A) = \phi(A, x_A, y_A, c_A, q_A).$$

Just as in § 23 we obtain the conditions

28.2 $$\phi[x, y, \mathfrak{J}(a, h), u] + \phi[y, h, \mathfrak{J}(a, x), u] + \phi[h, x, \mathfrak{J}(a, y), u] = 0$$
and
28.3 $$\phi[x, y, z, \mathfrak{J}(a, h)] + \phi[y, h, z, \mathfrak{J}(a, x)] + \phi[h, x, z, \mathfrak{J}(a, y)] = 0$$

which must be satisfied for all values of the normal vector a in addition to the differential condition

28.4 $$\phi'(x, y, z, u, h) + \phi'(y, h, z, u, x) + \phi'(h, x, z, u, y) = 0$$

where $\phi'(x, y, z, u, h)$ is the differential of $\phi(x, y, z, u)$.

Let us apply this to the Riemann tensor $\rho(x, y; z, u)$. The differential condition is always satisfied; in fact it is nothing but the Bianchi relation 12.5 (First paper) proved to hold in the general case toward the end of § 18 (Second paper). It is worth while to remember here that the differential condition for the indicator of a surface coincides with the Codazzi relation which thus plays for the indicator the same rôle as the Bianchi relation plays for the Riemann tensor. We need only consider *one* of the algebraic conditions 28.2 and 28.3 since the variables z and u enter in ρ in the same way (only the sign is changed when they are interchanged so that 28.2 is equivalent to 28.3 in this case). If we use expression 17.3 (Second paper) for the Riemann tensor, 28.2 becomes

$$28.4 \quad \begin{vmatrix} s[x, \mathfrak{F}(a,h)] & s(x,u) \\ s[y, \mathfrak{F}(a,h)] & s(y,u) \end{vmatrix} + \begin{vmatrix} s[y, \mathfrak{F}(a,x)] & s(y,u) \\ s[h, \mathfrak{F}(a,x)] & s(h,u) \end{vmatrix} + \begin{vmatrix} s[h, \mathfrak{F}(a,y)] & s(h,u) \\ s[x, \mathfrak{F}(a,y)] & s(x,u) \end{vmatrix} = 0.$$

Expanding the determinants we obtain a sum of six products, the first of which is $s[x, \mathfrak{F}(a,h)] . s(y,u)$; we may identify this with the last term of 23.5 considering $s(y,u)$ as a, x as k_A, and $\mathfrak{F}(a,h)$ as h_A; using then the form of the middle term of identity 23.5 we may write our product as $\mathfrak{F}(a,h) . \mathfrak{F}[s(y,u), x]$; applying this to all the six products of the expansion of 28.4 and changing the order of terms we may write it as

$$\begin{aligned} \mathfrak{F}(a,h) . \mathfrak{F}[s(y,u),x] - \mathfrak{F}(a,x) . \mathfrak{F}[s(y,u),h] + \mathfrak{F}(a,x) . \mathfrak{F}[s(h,u),y] \\ - \mathfrak{F}(a,y) . \mathfrak{F}[s(h,u),x] + \mathfrak{F}(a,y) . \mathfrak{F}[s(x,u),h] \\ - \mathfrak{F}(a,h) . \mathfrak{F}[s(x,u),y] = 0. \end{aligned}$$

This vanishes when condition 25.5 *is satisfied;* this is very natural because in this case the Riemann tensor gives the full infinitesimal variation of normals and therefore its integral over a piece of surface gives the finite variation of normals for the frontier of this piece, i. e. for the contour which limits this piece and, of course, this does not depend upon the surface—just as the integral of the indicator along a curve depends only on its end points. In the general case the Riemann tensor does not give the full infinitesimal variation, its integral does not give therefore the finite variation and there is no reason why it should be determined by the contour alone.

As our next example we take the contracted Riemann tensor which we write, according to § 26, as $\rho(x, I; z, I)$. This is a symmetric tensor of rank two which we denote by $\epsilon(x,z)$; but instead of considering it as such, we make of it a tensor of rank n: we substitute for one of its arguments, say z, the vector product of $n-1$ vectors. If we take, for instance, $n=4$ we have to make $z = \overline{uvw}$, i. e. the vector product of the vectors u, v, w (see end of § 26).

Now we have in $\epsilon(x, \overline{uvw})$ a tensor which is alternating in the 3 arguments u, v, w. It may be therefore integrated over a volume (3-dimensional) and gives a vector of the containing space; we shall see what are the integrability conditions, i. e. under what conditions the integral vanishes over every closed volume. If we write

$$\psi(u, v, w) = \epsilon(c_A, u, v, w)$$

the conditions in question are given by

$$\psi'(u, v, w, h) - \psi'(v, w, h, u) + \psi'(w, h, u, v) - \psi'(h, u, v, w) = 0.$$

This again gives a differential condition for ϵ viz.,

$$\epsilon'(x, \overline{uvw}, h) - \epsilon'(x, \overline{vwh}, u) + \epsilon'(x, \overline{whu}, v) - \epsilon'(x, \overline{huv}, w) = 0$$

which may be written as

$$\epsilon'(x, I, I) = 0$$

and an algebraic condition, viz.,

$$\epsilon\,[\mathfrak{J}(a, h), \overline{uvw}] - \epsilon\,[\mathfrak{J}(a, v), \overline{vwh}] + \epsilon\,[\mathfrak{J}(a, v), \overline{whu}] - \epsilon\,[\mathfrak{J}(a, w), \overline{huv}] = 0$$

or

$$\epsilon\,[\mathfrak{J}(a, I), I] = 0.$$

The discussion of spaces where these equations are satisfied will be given elsewhere.

§ 29. *Conclusion. Integral Expressions Which do Not Require the Consideration of the Containing Space.*

We had to go out of the space in order to consider integrals in the preceding sections as soon as the integrand had superfluous arguments, i. e. more than one in the case of a line integral, more than two (in which the tensor is antisymmetric) in the case of a surface integral, and so on. But it is desirable to be able to form in such cases also integral expressions not involving the conideration of the containing space. In some cases it is possible. Let us begin again with line integrals and let us take as the integrand a tensor field of the second rank which is symmetric in its arguments, so that we have

29.1 $$\phi(x, y) = \phi(y, x).$$

It can be proved that such a tensor can be presented in the form

29.2 $$\phi(x, y) = \pm (ax)(ay) \pm (bx)(by) \pm (cx)(cy) + \cdots$$

where a, b, c, $\cdots$ are mutually perpendicular vectors, the number of terms being equal to the number of dimensions of the space and the ± signs being used in order to avoid the introduction of imaginary vectors. Now the products ax, bx, cx, etc. are tensors of the first rank; they can be integrated, therefore, without leaving the curved space; we introduce the notations

$$P = \int a.dA, \qquad Q = \int b.dA, \qquad R = \int c.dA, \quad \cdots \tag{29.3}$$

and form the expression

$$\pm P^2 \pm Q^2 \pm R^2 \pm \cdots \tag{29.4}$$

using the same signs as in the corresponding places in 29.2. The expression 29.4 is the sought integral expression. Let us consider as an example the case of an ordinary (two-dimensional) surface and as the integrand let us take the indicator; if i, j denote the two unit vectors which belong to the principal directions and λ, μ are the curvatures we have $a = i\sqrt{\lambda}$, $b = j\sqrt{\mu}$ so that it will be convenient to introduce the notations $\sqrt{\lambda} = \alpha$, $\sqrt{\mu} = \beta$. Let us ask ourselves when the integrals P and Q are independent of the path. Since the integrand of P is $\omega(x) = \alpha.(ix)$ the condition 22.1 takes the form

$$\alpha'(h).(ix) + \alpha[i'(h).x] = \alpha'(x).(ih) + \alpha.[i'(x).h]; \tag{29.51}$$

for Q we have

$$\beta'(h).(jx) + \beta[j'(h).x] = \beta'(x).(jh) + \beta.[j'(x).h]. \tag{29.52}$$

On the other hand the Codazzi equation for the indicator, which now has the form

$$\alpha^2(ix)(iy) + \beta^2(jx)(jy),$$

gives

$$\begin{aligned} &2\alpha.\alpha'(h)(ix)(iy) + \alpha^2[i'(h).x](iy) + \alpha^2(ix)[i'(h).y] \\ &\quad + 2\beta.\beta'(h)(jx)(jy) + \beta^2[j'(h).x].(jy) + \beta^2(jx)[j'(h).y] \\ &= 2\alpha.\alpha'(x).(ih)(iy) + \alpha^2[i'(x).h](iy) + \alpha^2(ih)[i'(x).y] \\ &\quad + 2\beta.\beta'(x).(jh)(jy) + \beta^2[j'(x).h](jy) + \beta^2(jh)[j'(x).y]. \end{aligned} \tag{29.6}$$

The simultaneous existence of equations 29.51, 29.52 and 29.6 leads to the relation (see for the method used the author's paper in the January number of the *Transactions of the American Math. Society* for 1925, p. 123),

$$\alpha^2 + \beta^2 = 0 \quad \text{or} \quad \lambda + \mu = 0$$

so that only for a minimal surface are the integrals P and Q, and therefore the expression 29.4, independent of the path. These questions connected with minimal surfaces will be treated in detail elsewhere.

Let us consider now the case of a surface integral and a tensor field of rank four $\phi(x, y; u, v)$ with the property

$$\phi(x, y; u, v) = \phi(u, v; x, y)$$

as the integrand in a 4-S. Suppose that ϕ may be presented in the form

$$\lambda \begin{vmatrix} ix & iy \\ jx & jy \end{vmatrix} \cdot \begin{vmatrix} iu & iv \\ ju & jv \end{vmatrix} + \mu \begin{vmatrix} kx & ky \\ lx & ly \end{vmatrix} \cdot \begin{vmatrix} ku & kv \\ lu & lv \end{vmatrix}.$$

Precisely such a tensor occurs in physics, viz. the *second part* of the Riemann tensor of space-time. It is natural, in order to find an integral expression to put again $\sqrt{\lambda} = \alpha$, $\sqrt{\mu} = \beta$ and to consider the expression

$$P^2 + Q^2$$

where

$$P = \int \alpha \begin{vmatrix} i.dA & i.\delta A \\ j.dA & j.\delta A \end{vmatrix}, \quad Q = \int \beta \begin{vmatrix} k.dA & k.\delta A \\ l.dA & l.\delta A \end{vmatrix}.$$

This expression has in certain cases a physical meaning, being connected with the square of the electric charge; in other cases it is possible to obtain by a slight modification another more general integral expression of the same type. We shall not go into details concerning this type of integral expressions. The purpose of this section was only to indicate by way of conclusion, that the integrals involving the consideration of vectors of the containing space are not the only integral expressions which can be formed with tensor fields having superfluous arguments.

BALTIMORE, JUNE 14, 1925.

On Certain Symmetric Sums of Determinants.

By Lloyd L. Dines.

The first theorem of this paper has to do with a certain transformation of determinants which may be defined as follows. If D is a determinant of order n, and if $a_1, a_2, \cdots, a_n$ is any sequence of numbers (not necessarily distinct) each chosen from the set $0, 1, 2, \cdots, n$, then by the notation

$$T_{a_1a_2\ldots a_n} D$$

we shall mean the determinant of nth order obtained from D by shifting each element of the first column up a_1 places (thus eliminating the upper a_1 of them from the determinant), shifting each element of the second column up a_2 places, and so on to the nth column, and then filling in the vacant places at the bottom by zeros. From the definition it is obvious that $T_{00\ldots0}D \equiv D$, while if every a_i is positive or if any a_i is equal to n, then $D_{a_1a_2\ldots a_n}D = 0$. Between these extremes are the cases of interest.

For a given sequence $a_1, a_2, \cdots, a_n$, we shall consider all the distinguishable * permutations of the sequence.

By the notation

$$ST_{a_1a_2\ldots a_n} D$$

we shall mean the sum of all the determinants $T_{a_{i_1}a_{i_2}\cdots a_{i_n}}D$ where $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ is a permutation of $a_1, a_1, \cdots, a_n$. The theorem † to be proved in § 1 is then to the effect that *if at least one element of the sequence* $a_1, a_2, \cdots, a_n$ *is positive*

$$ST_{a_1a_2\ldots a_n}D \equiv 0.$$

For example for $n = 3$, we have

$$ST_{001}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & b_3 \\ b_1 & b_2 & c_3 \\ c_1 & c_2 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 & a_3 \\ b_1 & c_2 & b_3 \\ c_1 & 0 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_2 & a_3 \\ c_1 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{vmatrix} \equiv 0,$$

---

* If no two elements of the sequence are equal then all its permutations are distinguishable. If certain elements of the sequence are equal, permutations which differ only by the interchange of equal elements are not distinguishable. In this case only one representative of the set of non-distinguishable permutations will be considered.

† Theorems somewhat similar to this one and to its generalization stated in § 1 have been given by Sylvester, Hammond, and Deruyts. See Muir's *History of Determinants*, Vol. IV, pages 2 and 25.

and other vanishing sums corresponding to the sequences (0, 0, 2); (0, 1, 1); and (0, 1, 2).

The theorem of § 2 has to do with sums of determinants of nth order from a persymmetric * array of n rows, the elements however being affected by binomial coefficients * according to a definite law.

For $n=3$ the theorem would have reference to a persymmetric array

$$\begin{matrix} c_1 & c_2 & c_3 & \cdots \\ c_2 & c_3 & c_4 & \cdots \\ c_3 & c_4 & c_5 & \cdots \end{matrix}.$$

Denoting the binomial coefficients by the notation

$$\frac{m(m-1)\cdots(m-k+1)}{1\cdot 2\cdots k}=(m)_k,$$

we may state a simple case of the theorem as follows:

$$\sum_{j+k+l=p}\begin{vmatrix} (j+1)_1c_{j+1} & (k+2)_2c_{k+2} & (l+3)_3c_{l+3} \\ (j+2)_2c_{j+2} & (k+3)_3c_{k+3} & (l+4)_4c_{l+4} \\ (j+3)_3c_{j+3} & (k+4)_4c_{k+4} & (l+5)_5c_{l+5} \end{vmatrix}$$

$$=\sum_{j+k+l=p}(j+3)_3(k+4)_4(l+5)_5\begin{vmatrix} c_{j+1} & c_{k+2} & c_{l+3} \\ c_{j+2} & c_{k+3} & c_{l+4} \\ c_{j+3} & c_{k+4} & c_{l+5} \end{vmatrix},$$

the identity holding for every non-negative integer p.

§ 1.

Theorem: If D is a determinant of order n, and $a_1, a_2, \cdots, a_n$ *is a sequence of numbers (not necessarily all distinct) each chosen from the set* $0, 1, \cdots, n$, *and at least one of them positive, then*

$$ST_{a_1a_2\ldots a_n}D\equiv 0.$$

For $n=2$, the theorem is obviously true since

$$ST_{01}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\equiv\begin{vmatrix} a_{11} & a_{22} \\ a_{21} & 0 \end{vmatrix}+\begin{vmatrix} a_{21} & a_{12} \\ 0 & a_{22} \end{vmatrix}\equiv 0.$$

Assuming the theorem to be true for $n=r-1$, we will prove it true for $n = r$, thus establishing its general validity by mathematical induction.

---

* For literature on persymmetric determinants and on determinants whose elements are binomial coefficients, see Muir's *History of Determinants*, Vol. IV, Chapter XIII, and Vol. III, Chapter XX.

If we denote by $D^{(ij)}$ the algebraic complement of the element a_{ij} in the determinant

$$D \equiv | a_{ij} | \quad \begin{matrix} i = 1, 2, \cdots, r, \\ j = 1, 2, \cdots, r \end{matrix}$$

we find upon consideration of the expansion of D according to elements of its first row, that

$$(1) \qquad ST_{a_1a_2\ldots a_n} D \equiv \sum_{k=1}^{r}{}' \sum_{j=1}^{r} a_{1+a_k,j}\, ST_{a_1a_2\ldots a_{k-1}a_{k+1}\ldots a_r} D^{(1j)},$$

the notation $\sum_{k=1}^{r}{}'$ indicating that in case a_k has the same value for two or more values of k, the summation as to k shall include only one term representing such values for each value of j.

Now with respect to the symmetric sum S on the right side of (1), there are two possibilities. First, if any one of the numbers $a_1, a_2, \cdots, a_{k-1}, a_{k+1}, \cdots, a_r$ is different from zero, then the symmetric sum vanishes on account of our assumption that our theorem is true for determinants of order $r-1$. On the other hand, if $a_1 = a_2 = \cdots = a_{k-1} = a_{k+1} = \cdots = a_r = 0$ then by hypothesis $a_k \neq 0$. And for the value of k which gives rise to this situation, the right side of (1) reduces to

$$\sum_{j=1}^{r} a_{1+a_k,j}\, D^{(1j)},$$

which vanishes by an elementary theorem on determinants. Hence the entire right side of (1) vanishes, and our theorem is established.

A number of variations of the theorem immediately suggest themselves. For example, the "shifting" involved in the transformation T might obviously be made downward, to the right, or to the left, as well as upward. We will state one extension which is not quite so obvious, and which will be used in § 2.

With each element a_k of the sequence $a_1, a_2, \cdots, a_n$ we will associate a sequence of $n - a_k$ elements

$$(2) \qquad s_k = (s_{1k}, s_{2k}, \cdots, s_{n-a_k,k}) \quad (k = 1, 2, \cdots, n),$$

subject only to the restriction that if $a_i = a_j$, then $s_i \equiv s_j$. And we will denote by

$$T_{\substack{s_1s_2\ldots s_n \\ a_1a_2\ldots a_n}} D$$

the determinant obtained from D by shifting the elements of the kth column up a_k places, filling in the vacant places by zeros, and then multiplying the upper $n-a_k$ elements of the resulting column by the respective elements of the sequence s_k, this procedure to be followed for each column of D.

By the notation

$$ST_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}} D$$

we shall mean the sum of all determinants

$$T_{\substack{s_{i_1} s_{i_2} \cdots s_{i_n} \\ a_{i_1} a_{i_2} \cdots a_{i_n}}}$$

where $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ is a permutation of $a_1, a_2, \cdots, a_n$. We may then state the generalization in the form:

If D is a determinant of order n, and $a_1, a_2, \cdots, a_n$ is a sequence of numbers chosen from the set $0, 1, 2, \cdots, n$, of which not all elements are zero, and $s_1, s_2, \cdots, s_n$ are sequences of form (2); then

$$ST_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}} D \equiv 0.$$

The proof may be made by an induction similar to that used in the simpler theorem.

§ 2.

In this section, we suppose the elements of the persymmetric array

$$\begin{array}{ccccc} c_1 & c_2 & c_3 & \cdot & \cdot \\ c_2 & c_3 & c_4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & c_{n+2} & \cdot & \cdot \end{array}$$

to be well defined for as many columns as the statement of the theorem requires.

Denoting the binomial coefficients by the notation

$$(m)_k = \frac{m(m-1)\cdots(m-k+1)}{1\cdot 2\cdots k},$$

we will prove the following

Theorem: For any non-negative integer p, and any sequence of non-negative integers $r_1, r_2, \cdots, r_n$,

$$(3)\quad \sum_{j_1+j_2+\cdots+j_n=p} \begin{vmatrix} (j_1+r_1+1)_{r_1+1}c_{j_1+r_1+1} & (j_2+r_2+1)_{r_2+1}c_{j_2+r_2+1} & \cdots & (j_n+r_n+1)_{r_n+1}c_{j_n+r_n+1} \\ (j_1+r_1+2)_{r_1+2}c_{j_1+r_1+2} & (j_2+r_2+2)_{r_2+2}c_{j_2+r_2+2} & \cdots & (j_n+r_n+2)_{r_n+2}c_{j_n+r_n+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (j_1+r_1+n)_{r_1+n}c_{j_1+r_1+n} & (j_2+r_2+n)_{r_2+n}c_{j_2+r_2+n} & \cdots & (j_n+r_n+n)_{r_n+n}c_{j_n+r_n+n} \end{vmatrix}$$

$$= \sum_{j_1+j_2+\cdots+j_n=p} (j_1+r_1+n)_{r_1+n}(j_2+r_2+n)_{r_2+n}\cdots(j_n+r_n+n)_{r_n+n}$$

$$\times \begin{vmatrix} c_{j_1+r_1+1} & c_{j_2+r_2+1} & \cdots & c_{j_n+r_n+1} \\ c_{j_1+r_1+2} & c_{j_2+r_2+2} & \cdots & c_{j_n+r_n+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{j_1+r_1+n} & c_{j_2+r_2+n} & \cdots & c_{j_n+r_n+n} \end{vmatrix}$$

For the proof we shall make use of the following easily deducible relation between binomial coefficients:

$$(j+r+h)_{r+h}$$
$$\equiv (j+r+n)_{r+n}-(n-h)_1(j+r+n-1)_{r+n}+(n-h)_2(j+r+n-2)_{r+n} - \cdots \pm (j+r+h)_{r+n},$$

which holds for $h=1, 2, \cdots, n$.

By means of this relation, the elements of any column in (3) may be replaced respectively by polynomials

$$(j+r+n)_{r+n}c_{j+r+1}-(n-1)_1(j+r+n-1)_{r+n}\,c_{j+r+1}+\cdots \pm (j+r+1)_{r+n}c_{j+r+1}$$
$$(j+r+n)_{r+n}c_{j+r+2}-(n-2)_1(j+r+n-1)_{r+n}\,c_{j+r+2}+\cdots \mp (j+r+2)_{r+n}c_{j+r+2}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(j+r+n)_{r+n}c_{j+r+n-1}-(1)_1(j+r+n-1)_{r+n}c_{j+r+n-1}$$
$$(j+r+n)_{r+n}c_{j+r+n}$$

the symbols j and r carrying subscripts indicative of the particular column considered.

Each determinant of the sum (3) can therefore be replaced by the sum of n^n constituent determinants (many of which obviously vanish) with the property that in each column all the elements contain a common binomial coefficient factor. The typical form of these columns will be

$$
(4) \qquad \begin{array}{c}
(-1)^a (n-1)_a (j+r+n-a)_{r+n} c_{j+r+1} \\
(-1)^a (n-2)_a (j+r+n-a)_{r+n} c_{j+r+2} \\
\cdot \\
\cdot \\
\cdot \\
(-1)^a (a)_a (j+r+n-a)_{r+n} c_{j+r+n-a} \\
0 \\
\cdot \\
\cdot \\
0
\end{array}
$$

the last a elements being zero. There will be such columns corresponding to $a = 0, 1, \cdots, n-1$, and we will say that the column (4) has the *character* a.

Consider now any set of numbers $a_1, a_2, \cdots, a_n$, each chosen from the sequence $0, 1, \cdots, n-1$, not necessarily distinct, but at least one of them being zero. Consider the constituent determinant in which the first column (involving the index j_1) has the character a_1, the second column (involving the index j_2) has the character a_2, and so on. We will say that this constituent determinant has the character $(a_1, a_2, \cdots, a_n)$. Out of this determinant we can take the factor

$$
(5) \qquad (-1)^{a_1+a_2+\cdots+a_n} (j_1+r_1+n-a_1)_{r_1+n} (j_2+r_2+n-a_2)_{r_2+n} \cdots (j_n+r_n+n-a_n)_{r_n+n},
$$

leaving a residual determinant of which a typical column is

$$
\begin{array}{c}
(n-1)_a c_{j+r+1} \\
(n-2)_a c_{j+r+2} \\
\cdot \\
\cdot \\
\cdot \\
(a)_a c_{j+r+n-a} \\
0 \\
\cdot \\
\cdot \\
0
\end{array}
$$

If $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ is any permutation of $(a_1, a_2, \cdots, a_n)$ there will be a constituent determinant having the character $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$, and out of it we can take the factor

$$(6)\quad (-1)^{a_1+a_2+\cdots+a_n}(j_1+r_1+n-a_{i_1})_{r_1+n}(j_2+r_2+n-a_{i_2})_{r_2+n}\cdots (j_n+r_n+n-a_{i_n})_{r_n+n},$$

which in general will be different from (5).

It now becomes necessary for us to note a significant property of the sum (3) which we have not previously stressed, and which will enable us to correlate the products (5) and (6). The notation

$$\sum_{j_1+j_2+\cdots+j_n=p}$$

indicates that the summation is to be made for every combination of indices $j_1, j_2, \cdots, j_n$ of which the sum is p. Hence if in a determinant affected by this summation sign we make a transformation of indices

$$j_1 = j'_1 + a_1, \quad j_2 = j'_2 + a_2, \cdots, \quad j_n = j'_n + a_n,$$

there will be no change in the content of the summation, *provided* $a_1 + a_2 + \cdots + a_n = 0$.

If therefore in the determinant with character $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ we make the substitution

$$j_1 = j'_1 + a_{i_1} - a_1, \quad j_2 = j'_2 + a_{i_2} - a_2, \cdots, \quad j_n = j'_n + a_{i_n} - a_n,$$

the content of the summation will not be affected. And upon dropping the primes from the indices $j'_1, j'_2, \cdots, j'_n$ in the product (6) after transformation it is seen to be identical with the product (5).

By means of similar transformations we are able to reduce each determinant whose character is a permutation of $(a_1, a_2, \cdots, a_n)$ to a form such that the identical factor (5) can be taken from it.

It is not very difficult to see further that the residual determinants are obtainable by a transformation $T_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}}$ from one and the same determinant. We may in fact write the sum (as indicated by $\sum_{j_1-j_2+\cdots+j_n=p}$) of all the constituent determinants whose characters are permutations of $(a_1, a_2, \cdots, a_n)$ in the form

$$(7)\quad \sum_{j_1+j_2+\cdots+j_n=p}(-1)^{a_1+a_2+\cdots+a_n}(j_1+r_1+n-a_1)_{r_1+n}(j_2+r_2+n-a_2)_{r_2+n}\cdots (j_n+r_n+n-a_n)\, ST_{\substack{s_1 s_2 \cdots s_n \\ a_1 a_2 \cdots a_n}} D,$$

where

$$D = \begin{vmatrix} c_{j_1+r_1+1-a_1} & c_{j_2+r_2+1-a_2} & \cdot & \cdot & c_{j_n+r_n+1-a_n} \\ c_{j_1+r_1+2-a_1} & c_{j_2+r_2+2-a_2} & \cdot & \cdot & c_{j_n+r_n+2-a_n} \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ c_{j_1+r_1+n-a_1} & c_{j_2+r_2+n-a_2} & \cdot & \cdot & c_{j_n+r_n+n-a_n} \end{vmatrix}$$

and

$$s_k = ((n-1)_{a_k},\ (n-2)_k,\ \cdots,\ (a_k)_{a_k}).$$

We are now ready to draw our conclusion. If the set $(a_1, a_2, \cdots, a_n)$ contains an element different from zero, the sum (7) vanishes by the generalized theorem of § 1. Therefore the only non-vanishing part of the sum (3) is given by (7) for $a_1 = a_2 = \cdots = a_n = 0$. This value is evidently

$$\sum_{j_1+j_2+\,.\,.\,+j_n=p} (j_1+r_1+n)_{r_1+n}(j_2+r_2+n)_{r_2+n}\cdots(j_n+r_n+n)_{r_n+n}$$
$$\times \begin{vmatrix} c_{j_1+r_1+1} & c_{j_2+r_2+1} & \cdot\ \cdot & c_{j_n+r_n+1} \\ c_{j_1+r_1+2} & c_{j_2+r_2+2} & \cdot\ \cdot & c_{j_n+r_n+2} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ c_{j_1+r_1+n} & c_{j_2+r_2+n} & \cdot\ \cdot & c_{j_n+r_n+n} \end{vmatrix}$$

and this is the value as stated in the theorem.

UNIVERSITY OF SASKATCHEWAN.

A Direct Solution of Systems of Linear Differential Equations having Constant Coefficients.*

BY J. A. NYSWANDER.

1. *Introduction.* The theory of linear differential equations having constant coefficients began with Euler, about the middle of the eighteenth century, and was continued during the latter part of the century by other mathematicians, the most noteworthy of whom were Gauss, Laplace, and Lagrange. All of these writers except Lagrange treated a single equation of low order; Lagrange, however, was able to solve those special systems of linear differential equations in a few dependent variables for which the roots of the characteristic equation $D(\lambda)=0$ were distinct. During the interval of about seventy years between Lagrange and Weierstrass, various symbolic methods were devised for solving a single equation of the nth order, while for the general simultaneous system of equations a treatment was given only for the simple case in which $D(\lambda)=0$ has distinct roots. In this period special systems of equations for which $D(\lambda)=0$ has multiple roots were solved by regarding them as the limiting cases of corresponding systems of equations for which $D(\lambda)=0$ has distinct roots. This method is unsatisfactory logically and practically impossible of application.

The first complete treatment of a system of differential equations in n dependent variables was given by Weierstrass.† He started with the system

$$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i=1,\cdots,n), \tag{1}$$

where $x'_i \equiv dx_i/dt$, and made linear transformations on certain bilinear forms obtained from (1), thus securing a simplified equivalent system of equations of the form

$$\left\{\begin{array}{l} \xi'_1 = \lambda_1\xi_1, \\ \xi'_2 = \lambda_1\xi_2 + \delta_2\xi_1, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \xi'_j = \lambda_1\xi_j + \delta_j\xi_{j-1}, \end{array}\right. \tag{2}$$

---

[*] Read before the American Mathematical Society at Chicago, April 18, 1924.

[†] Weierstrass, *Collected Works,* Vol. II, pp. 74-75.

where λ_i is a root of order j of the equation $D(\lambda)=0$. The properties of $D(\lambda)$ pertinent to the problem were expressed in terms of the elementary divisors associated with $D(\lambda)$ and its minors. Although Weierstrass' process is logically sound it is indirect and tedious in application. Solutions of the problem were also given by Weber [*] and Frobenius.[†] Their methods are essentially the same in principle as that of Weierstrass and differ from it only in certain minor details.

A method of treatment [‡] which is entirely different in character from that of Weierstrass was given by Thomé. By differentiating a single one of the differential equations $n-1$ times and replacing the x'_i at each step by $\sum_{j=1}^{n} a_{ij}x_j$, Thomé arrived at a single differential equation of the nth order in one of the x_i, say x_1. This procedure implies that a certain determinant of the coefficients of $x_2, \cdots, x_n$ in the n differentiated equations is different from zero. Each of the n solutions of this differential equation of the nth order gives rise to a set of simultaneous linear equations in $x_2, \cdots, x_n$ which can be solved. If, however, this determinant is zero the procedure just indicated leads to a system of non-homogeneous differential equations which must be solved in the process of arriving at the solutions of system (1). Thomé's treatment, although simpler in theory and application than any of the earlier methods, is likewise indirect and laborious to apply.

The present paper treats the general system of differential equations by a method which is both elementary and direct, and which at the same time is much simpler of application than any of the previous methods. A general form for the solution of system (1) is obtained, from which, in any particular case, the desired solutions can be written at once in terms of the minors of the determinant $D(\lambda_i)$ and of their λ-derivatives.

2. *The Solutions when* $D(\lambda)=0$ *has Distinct Roots.* Before solving the problem in its greatest degree of generality the two cases for which the characteristic equation $D(\lambda)=0$ has distinct roots, and a triple root, respectively, will be treated.

Consider the simultaneous system of equations

$$(1) \qquad x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i=1, \cdots, n),$$

---

[*] H. Weber, *Lehrbuch der Algebra*, Vol. II, pp. 41-42.

[†] Crelle's *Journal*, Vol. 86, pp. 42-72; 146-209.

[‡] Crelle's *Journal*, Vol. 131, pp. 8-24; Vol. 133, pp. 1-17.

where the a_{ij} are given constants. Let the question be raised whether or not they have a solution of the form

$$x_i = A_i e^{\lambda t} \qquad (i = 1, \cdots, n), \tag{2}$$

in which the A_i and λ are constants. On substituting (2) in (1), it is found that if (2) is a solution, then the A_i and λ must satisfy the conditions

$$\left\{\begin{array}{l} (a_{11} - \lambda) A_1 + a_{12} A_2 + \cdots + a_{1n} A_n = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1} A_1 + a_{n2} A_2 + \cdots + (a_{nn} - \lambda) A_n = 0. \end{array}\right. \tag{3}$$

Equations (3) are linear in the A_i and will have a common solution in which not all A_i are zero if, and only if, λ is a root of the characteristic equation

$$D(\lambda) = \begin{vmatrix} a_{11} - \lambda, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22} - \lambda, & \cdots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \cdots, & a_{nn} - \lambda \end{vmatrix} = 0. \tag{4}$$

Suppose that $D(\lambda) = 0$ has n distinct roots $\lambda_1, \cdots, \lambda_n$. For each of these n roots equations (3) have a solution for which not all the A_i are zero. Since for each $\lambda = \lambda_j$ at least one first minor of $D(\lambda_j)$ does not vanish, one can express the n solutions of (3) in the form

$$A_j = \delta_{j1i} c_i \qquad (j, i = 1, \cdots, n), \tag{5}$$

where the c_i are arbitrary constants and the δ_{j1i} are the first-minor cofactors in $D(\lambda_i)$ resulting from the solution of (3) for the A_j by determinants. In general, the second subscript k of δ_{jki} is used in indicate that δ_{jki} represents a kth minor cofactor (of order $n - k$) of the determinant $D(\lambda_i)$.

On substituting (5) in (2), one obtains the n solution of system (1)

$$\left\{\begin{array}{l} x_1 = c_1 \delta_{11i} e^{\lambda_i t} \qquad (i = 1, \cdots, n), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = c_i \delta_{n1i} e^{\lambda_i t}. \end{array}\right. \tag{6}$$

A proof of the fact that (6) form a fundamental set of solutions of (1) will be omitted here since a demonstration of this property for the general case will be given in Section 7.

3. *Solutions of System* (1) *when* $D(\lambda) = 0$ *has a Triple Root.* Let $\lambda_1, \lambda_4, \cdots, \lambda_n$ be the n roots of $D(\lambda) = 0$, $\lambda = \lambda_1$ being a triple root. The following cases, classified with respect to the properties of $D(\lambda_1)$, can arise:

I. All first and second-minor cofactors (δ_{j1k} and δ_{j2k}) of $D(\lambda_1)$ are zero, while at least one third-minor cofactor (δ_{j3k}) is not zero.
II. At least one δ_{i1k} is not zero.
III. All δ_{i1k} are zero, at least one δ_{i2k} is not zero, and at least one $\delta_{i1k}(\lambda)$ has a λ-derivative that does not vanish for $\lambda = \lambda_1$.

Since the λ-derivative of every $(j-1)$st minor of $D(\lambda)$ is a sum of jth minors of $D(\lambda)$, it is evident that

$$\left[\frac{d^3D(\lambda)}{d\lambda^3}\right]_{\lambda=\lambda_1} = \sum_{i,\,k} \delta_{i3k}.$$

Thus if $\lambda = \lambda_1$ is a triple root of $D(\lambda) = 0$, the case for which all $\delta_{i1k} =$ all $\delta_{i2k} =$ all $\delta_{i3k} = 0$ cannot exist. It would appear that under III there would arise the sub-case obtained by replacing the last hypothesis of III by the one that every

$$\left[\frac{d}{d\lambda}\,\delta_{i1k}(\lambda)\right]_{\lambda=\lambda_1} = 0\ ,$$

at least one

$$\left[\frac{d}{d\lambda}\,\delta_{i2k}(\lambda)\right]_{\lambda=\lambda_1} \neq 0.$$

That this cannot occur when $\lambda = \lambda_1$ is only a triple root of $D(\lambda) = 0$ is readily seen by considering $D(\lambda)$ when reduced to its normal form.*

CASE I. In Section 2 it was seen that if the system of differential equations

(1) $$x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n),$$

has a solution of the form

(2) $$x_i = A_i e^{\lambda_1 t} \qquad (i = 1, \cdots, n),$$

then the A_i must satisfy the equations

(3) $$(a_{ii} - \lambda_1)A_i + \sum_{k=1}^{n} a_{ik}A_k = 0 \qquad (i = 1, \cdots, n;\ k \neq i).$$

Since by the hypothesis of this case $D(\lambda_1)$ is of rank $n - 3$, the solution of (3) may be written in the form

(4) $$A_i = \delta_{i31}c_1 + \delta_{i32}c_2 + \delta_{i33}c_3 \qquad (i = 1, \cdots, n),$$

where c_1, c_2 and c_3 are arbitrary constants, and the δ_{i3k} are the third-minor cofactors in $D(\lambda_1)$ obtained in the solution of system (3) by determinants.

---

* Bôcher, *Introduction to Higher Algebra*, p. 267.

One observes that in consequence of the nature of solution (4), at least two of the δ_{i3k} $(i=1,\cdots,n)$ are identically zero for $k=1, 2, 3$ respectively, while at least one of the δ_{i3k}, in each of these three sets, is different from zero. On substituting (4) in (2), one obtains as the three solutions of equations (1)

$$x_j = c_i\delta_{j3i}e^{\lambda_1 t} \qquad (j=1,\cdots,n;\ i=1, 2, 3). \tag{5}$$

The remaining $n-3$ solutions of system (1) corresponding to $\lambda=\lambda_4,\cdots,\lambda_n$ are given by (6), of Section 2, for $i=4,\cdots,n$ respectively.

Case II. At least one δ_{i1k} is not zero. It will be shown that in this case the solutions of equations (1), associated with $\lambda=\lambda_1$, have the form

$$x_i = (A_i^{(1)}t^2 + A_i^{(2)}t + A_i^{(3)})e^{\lambda_1 t} \qquad (i=1,\cdots,n), \tag{6}$$

where the $A_i^{(j)}$ are constants. On substituting (6) in (1) and equating coefficients of t^2, t and t^0, one obtains as sufficient conditions that (6) shall be a solution of (1)

$$(a_{ii}-\lambda_1)A_i^{(1)} + \sum_{k=1}^{n} a_{ik}A_k^{(1)} = 0 \qquad (i=1,\cdots,n;\ k\neq i), \tag{7}$$

$$(a_{ii}-\lambda_1)A_i^{(2)} + \sum_{k=1}^{n} a_{ik}A_k^{(2)} = 2A_i^{(1)} \qquad (i=1,\cdots,n;\ k\neq i), \tag{8}$$

$$(a_{ii}-\lambda_1)A_i^{(3)} + \sum_{k=1}^{n} a_{ik}A_k^{(3)} = A_i^{(2)} \qquad (i=1,\cdots,n;\ k\neq i), \tag{9}$$

It will now be shown that these conditions can be satisfied.

In order to obtain the solutions of (7), (8), and (9), start with the identities

$$\left\{\begin{array}{l} (a_{11}-\lambda)\delta_{11}(\lambda) + \cdots + a_{1n}\delta_{n1}(\lambda) \underset{\lambda}{\equiv} 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n-1,1}\delta_{11}(\lambda) + \cdots + a_{n-1,n}\delta_{n1}(\lambda) \underset{\lambda}{\equiv} 0, \\ a_{n1}\delta_{11}(\lambda) + \cdots + (a_{nn}-\lambda)\delta_{n1}(\lambda) \underset{\lambda}{\equiv} D(\lambda), \end{array}\right. \tag{10}$$

where $\delta_{11}(\lambda),\cdots,\delta_{n1}(\lambda)$ are the cofactors of the elements $a_{n1},\cdots,a_{nn}-\lambda$ respectively in $D(\lambda)$. The third subscript of the δ_{j1k}, which indicates the value of λ under consideration, will be omitted except when it is necessary to distinguish notationally between several groups of cofactors of the same order. In writing the identities (10) it has been assumed, for convenience, that the first $n-1$ rows of $D(\lambda_1)$ contain a non-vanishing first-minor.

On differentiating (10) twice with respect to λ and setting $\lambda = \lambda_1$, it is found that the relations

$$(11) \qquad (a_{ii} - \lambda_1)\delta_{i1} + \sum_{k=1}^{n} a_{ik}\delta_{k1} = 0 \qquad (i = 1, \cdots, n;\ k \neq i),$$

$$(12) \qquad (a_{ii} - \lambda_1)\delta'_{i1} + \sum_{k=1}^{n} a_{ik}\delta'_{k1} = \delta_{i1} \qquad (i = 1, \cdots, n;\ k \neq i),$$

$$(13) \qquad (a_{ii} - \lambda_1)\delta''_{i1} + \sum_{k=1}^{n} a_{ik}\delta''_{i1} = 2\delta'_{i1} \qquad (i = 1, \cdots, n;\ k \neq i),$$

are satisfied. In relations (12) and (13) the notation

$$\delta'_{ik} = \left[\frac{d}{d\lambda}\ \delta_{ik}(\lambda)\right]_{\lambda=\lambda_1}, \qquad \delta''_{ik} = \left[\frac{d^2}{d\lambda^2}\ \delta_{ik}(\lambda)\right]_{\lambda=\lambda_1}, \qquad \delta_{ik}^{(a)} = \left[\frac{d^a}{d\lambda^a}\ \delta_{ik}(\lambda)\right]_{\lambda=\lambda}$$

has been used. Relations (11), (12), and (13) give, at once, the following solutions of equations (7), (8), and (9) respectively:

$$(14) \qquad A_i^{(1)} = c_1\delta_{i1} \qquad (i = 1, \cdots, n),$$

$$(15) \qquad A_i^{(2)} = 2c_1\delta'_{i1} + c_2\delta_{i1} \qquad (i = 1, \cdots, n),$$

$$(16) \qquad A_i^{(3)} = c_1\delta''_{i1} + c_2\delta'_{i1} + c_3\delta_{i1} \qquad (i = 1, \cdots, n),$$

where c_1, c_2, and c_3 are arbitrary constants. On substituting in (13) the values of the $A_i^{(j)}$ just determined, one obtains three solutions of (1) which can be written in the following form:

$$(17) \qquad x_i = c_1(\delta''_{i1} + 2\delta'_{i1}t + \delta_{i1}t^2)e^{\lambda_1 t} + c_2(\delta'_{i1} + \delta_{i1}t)e^{\lambda_1 t} + c_3\delta_{i1}e^{\lambda_1 t} \qquad (i = 1, \cdots, n).$$

The remaining $n - 3$ solutions of (1) are given by (6), of Section 2.

CASE III. All δ_{i1k} are zero, at least one δ_{i2k} is not zero, and at least one δ'_{i1k} is different from zero.

By the procedure employed in Case I, one obtains the two purely exponential solutions of (1)

$$(18) \qquad x_i = c_1\delta_{i21}e^{\lambda_1 t} + c_2\delta_{i22}e^{\lambda_1 t} \qquad (i = 1, \cdots, n).$$

The remaining solution of System (1) corresponding to $\lambda = \lambda_1$ is obtained from the first of the three solutions included in (17). For the hypotheses of Case III this solution becomes

$$(19) \qquad x_i = c_3(\delta''_{i1} + 2\delta'_{i1}t)e^{\lambda_1 t} \qquad (i = 1, \cdots, n).$$

The remaining $n - 3$ solutions of (1) are given by (6) of Section 2.

4. *Fundamental Sets of Solutions.* For each of the cases considered in the two preceding articles n solutions of equations (1) were obtained, but it was not shown that they form a fundamental set. Although this property will be established in Section 7 for every case that can arise, yet it seems advisable at this point to illustrate the method by proving the independence of the n solutions in Case II, of Section 3.

The n solutions of the system of differential equations (1) will form a fundamental set if, and only if, their determinant $\Delta(t)$ is different from zero. Suppose that the solutions

$$\begin{aligned} x_j = c_1(\delta''_{j1} + 2\delta'_{j1}t + \delta_{j1}t^2)e^{\lambda_1 t} + c_2(\delta'_{j1} + \delta_{j1}t)e^{\lambda_1 t} \\ + c_3\delta_{j1}e^{\lambda_1 t} + \sum_{i=4}^{n} c_i\delta_{j1i}e^{\lambda_i t} \qquad (j = 1, \cdot \cdot \cdot, n), \end{aligned} \tag{20}$$

do not form a fundamental set, and hence that $\Delta(t) = 0$. One easily verifies that

$$\Delta(t) = \Delta(0)e^{(a_{11} + \cdots + a_{nn})t},$$

where

$$\Delta(0) = \begin{vmatrix} \delta''_{11}, & \delta'_{11}, & \delta_{11}, & \delta_{114}, & \cdot \cdot \cdot, & \delta_{11n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta''_{n1}, & \delta'_{n1}, & \delta_{n1}, & \delta_{n14}, & \cdot \cdot \cdot, & \delta_{n1n} \end{vmatrix} = 0, \tag{21}$$

in consequence of the assumption $\Delta(t) = 0$.

The assumption (21) implies that there exist n constants $c_1^{(0)}, \cdot \cdot \cdot, c_n^{(0)}$ not all of which are zero, such that

$$\begin{aligned} c_1^{(0)}\delta''_{j1} + c_2^{(0)}\delta'_{j1} + c_3^{(0)}\delta_{j1} + c_4^{(0)}\delta_{j14} + \cdot \cdot \cdot + c_n^{(0)}\delta_{j1n} = 0 \\ (j = 1, \cdot \cdot \cdot, n). \end{aligned} \tag{22}$$

For this special set of values of the c_i, equations (20) become

$$\left\{ \begin{aligned} x_j = c_1^{(0)}(\delta''_{j1} + 2\delta'_{j1}t + \delta_{j1}t^2)e^{\lambda_1 t} + c_2^{(0)}(\delta'_{j1} + \delta_{j1}t)e^{\lambda_1 t} \\ + c_3^{(0)}\delta_{j1}e^{\lambda_1 t} + \sum_{i=4}^{n} c_i\delta^{(0)}_{j1i}e^{\lambda_i t} \qquad (j = 1, \cdot \cdot \cdot, n). \end{aligned} \right. \tag{23}$$

Thus (23) is a solution of (1) having, from (22), the initial conditions $x_1(0) = x_2(0) = \cdot \cdot \cdot = x_n(0) = 0$. The trivial solution $x_1 \equiv x_2 \equiv \cdot \cdot \cdot \equiv x_n \underset{t}{\equiv} 0$ also has the foregoing initial conditions. In view of the uniqueness of the solutions of (1) under given initial conditions, it follows that the x_j of (23) are identically zero in t.

Take any $x_j \equiv 0$ of (23) and differentiate the identity $n - 1$ times with respect to t. On setting $t = 0$ in each of these n identities, one obtains the n equations

$$(24)\quad \begin{cases} 0\delta''_{j1}c_1^{(0)} + 0\delta'_{j1}c_2^{(0)} + 1\delta_{j1}c_3^{(0)} + 1\delta_{j14}c_4^{(0)} + \cdots + 1\delta_{j1n}c_n^{(0)} = 0, \\ 0\delta''_{j1}c_1^{(0)} + 1\delta'_{j1}c_2^{(0)} + \lambda_1\delta_{j1}c_3^{(0)} + \lambda_4\delta_{j14}c_4^{(0)} + \cdots + \lambda_n\delta_{j1n}c_n^{(0)} = 0, \\ 2\delta''_{j1}c_1^{(0)} + 2\lambda_1\delta'_{j1}c_2^{(0)} + \lambda_1^2\delta_{j1}c_3^{(0)} + \lambda^2_4\delta_{j14}c_4^{(0)} + \cdots + \lambda^2_n\delta_{j1n}c_n^{(0)} = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (n-1)(n-2)\lambda_1^{n-3}\delta''_{j1}c_1^{(0)} + (n-1)\lambda_1^{n-2}\delta'_{j1}c_2^{(0)} + \lambda_1^{n-1}\delta_{j1}c_3^{(0)} \\ \qquad\qquad + \lambda_4^{n-1}\delta_{j14}c_4^{(0)} + \cdots + \lambda_n^{n-1}\delta_{j1n}c_n^{(0)} = 0, \end{cases}$$

which are linear and homogeneous in $c_1^{(0)}, \cdots, c_n^{(0)}$. Since not all of the $c_i^{(0)}$ are zero, it follows that the determinant Γ of the coefficients must satisfy the condition

$$(25)\quad \Gamma = \delta''_{j1}\delta'_{j1}\delta_{j1}\delta_{j14}\cdots\delta_{j1n} \begin{vmatrix} 0, & 0, & 1, & \cdots, & 1 \\ 0, & 1, & \lambda_1, & \cdots, & \lambda_n \\ 2, & 2\lambda_1, & \lambda_1^2, & \cdots, & \lambda_n^2 \\ 6\lambda_1, & 3\lambda_1^2, & \lambda_1^3, & \cdots, & \lambda_n^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (n-1)(n-2)\lambda_1^{n-3}, & (n-1)\lambda_1^{n-2}, & \lambda_1^{n-1}, & \cdots, & \lambda_n^{n-1} \end{vmatrix} =$$

which can be written in the convenient form

$$(26)\qquad \Gamma = \delta''_{j1}\delta'_{j1}\delta_{j1}\delta_{j14}\cdots\delta_{j1n}\Gamma_1 = 0.$$

That the determinant Γ_1 is different from zero is easily shown, for, let $\Lambda(\lambda)$ be the determinant obtained by replacing the elements of the first column of Γ_1 by $1, \lambda, \lambda^2, \cdots, \lambda^{n-1}$ respectively. The equation $\Lambda(\lambda) = 0$ of degree $n-1$ in λ has $\lambda_1, \lambda_1, \lambda_4, \cdots, \lambda_n$ as its $n-1$ roots. Consequently

$$\left[\frac{d^2\Lambda(\lambda)}{d\lambda^2}\right]_{\lambda=\lambda_1} = \Gamma_1 \neq 0,$$

for otherwise $\lambda = \lambda_1$ would be a triple root of $\Lambda(\lambda) = 0$. Since $\Gamma_1 \neq 0$ at least one of the factors in (26), multiplying Γ_1, must be zero. On omitting the corresponding column and considering the first $n-1$ equations in (24), one proves similarly that a second of the factors of (26) must be zero. By repeating this procedure one ultimately shows that

$$\delta''_{j1} = \delta'_{j1} = \delta_{j1} = \delta_{j14} = \cdots = \delta_{j1n} = 0 \qquad (j = 1, \cdots, n).$$

This result contradicts the fact that in each of the n solutions of (1) there is at least one non-zero coefficient. Therefore the assumption that $\Delta(t) = 0$

is false, and hence the n solutions of Case II, Section 3, form a fundamental set of solutions of equations (1).

5. *Derivation of a General Form of Solution for System* (1). It will now be shown that when $D(\lambda_1)$ and its minors have certain properties, the system of differential equations

$$(1) \qquad x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n),$$

has a solution of the form

$$(2) \qquad x_i = (A_i^{(1)}t^{\omega} + A_i^{(2)}t^{\omega-1} + \cdots + A_i^{(\omega)}t + A_i^{(\omega+1)})e^{\lambda_1 t} \qquad (i = 1, \cdots, n),$$

where the $A_i^{(j)}$ are constants and λ_1 is a root of the characteristic equation $D(\lambda) = 0$. On substituting (2) in (1) and equating coefficients of $t^{\omega}, \cdots, t^0$ one obtains as sufficient conditions for the existence of a solution of the form (2) the $\omega + 1$ sets of equations

$$(3) \qquad (a_{ii} - \lambda_1)A_i^{(1)} + \sum_{k=1}^{n} a_{ik}A_k^{(1)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i),$$

$$(4) \qquad (a_{ii} - \lambda_1)A_i^{(2)} + \sum_{k=1}^{n} a_{ik}A_k^{(2)} = \omega A_i^{(1)} \qquad (i = 1, \cdots, n;\ k \neq i),$$

. . . .

$$(5) \qquad (a_{ii} - \lambda_1)A_i^{(h)} + \sum_{k=1}^{n} a_{ik}A_k^{(h)} = (\omega - h + 2)A_i^{(h-1)} \qquad (i = 1, \cdots, n;\ k \neq i),$$

. . . .

$$(6) \qquad (a_{ii} - \lambda_1)A_i^{(\omega+1)} \sum_{k=1}^{n} a_{ik}A_k^{(\omega+1)} = A_i^{(\omega)} \qquad (i = 1, \cdots, n;\ k \neq i),$$

The solutions of these $\omega + 1$ sets of equations will be expressed in terms of the cofactors of $D(\lambda_1)$ and their λ-derivatives, and consequently it will be necessary to consider in detail the determinant

$$D(\lambda) = \left| \begin{array}{cccc|c|cc}
a_{11} - \lambda, & a_{12}, & \cdots, & a_{1,n-j}, & a_{1,n-j+1}, & \cdots, & a_{1n} \\
a_{21}, & a_{22} - \lambda, & \cdots, & a_{2,n-j}, & a_{2,n-j+1}, & \cdots, & a_{2n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{n-j,1}, & a_{n-j,2}, & \cdots, & a_{n-j,n-j} - \lambda, & a_{n-j,n-j+1}, & \cdots, & a_{n-j,n} \\
\cline{1-4}
a_{n-j+1,1}, & a_{n-j+1,2}, & \cdots, & \cdots, & a_{n-j+1,n-j+1} - \lambda, & \cdots, & a_{n-j+1,n} \\
\cline{1-5}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{n1}, & a_{n2}, & \cdot & \cdot & \cdot & \cdot\,, & a_{nn} - \lambda
\end{array} \right|$$

where the jth and $(j-1)$st principal minors, of orders $n-j$ and $n-j+1$ respectively, have been set off with lines because of the important role that they play in the subsequent theory. On referring to $D(\lambda)$ one readily verifies the identities

$$(7)\quad \begin{cases} (a_{ii}-\lambda)\delta_{ij}(\lambda)+\sum\limits_{k=1}^{n} a_{ik}\delta_{kj}(\lambda) \underset{\lambda}{\equiv} 0 \quad (i=1,\cdots,n-j;\ k\neq i), \\ (a_{ii}-\lambda)\delta_{ij}(\lambda)+\sum\limits_{k=1}^{n} a_{ik}\delta_{kj}(\lambda) \underset{\lambda}{\equiv} \delta_{i,j-1}(\lambda)\,(i=n-j+1,\cdots,n;\ k\neq i \end{cases}$$

where the $\delta_{ij}(\lambda)$ $(i=1,\cdots,n-j+1)$ are the jth minor cofactors, respectively (of order $n-j$), of the elements $a_{n-j+1,i}$ $(i=1,\cdots,n-j+1)$ in the upper left $(j-1)$st minor of $D(\lambda)$; and the $\delta_{i,j-1}(\lambda)$ $(i=n-j+1,\cdots,n)$ are the upper left $(j-1)$st minors of $D(\lambda)$ obtained by replacing the $(n-j+1)$st row by the $(n-j+1)$st, $(n-j+2)$nd, $\cdots$, nth rows, respectively.

For the foregoing particular choice in the position of the $(j-1)$st minor cofactors of $D(\lambda)$ occurring in (7) the δ_{kj} $(k=n-j+2,\cdots,n)$ are identically zero, by definition, and have been written for the purpose of securing generality in notation and logic, and also for the purpose of making the coefficients in (7) agree exactly, in form, with the coefficients in the left members of equations (3), $\cdots$, (6). This specialization in the position, in $D(\lambda)$, of the $(j-1)$st minors in question has been employed solely for the purpose of attaining simplicity in presentation; all subsequent work, however, is entirely general and applies to any minor-cofactors of $D(\lambda)$ satisfying certain conditions which will now be specified.

Suppose (a), that $(\lambda-\lambda_1)^s$ is a factor of every $(j-1)$st minor-cofactor (of order $n-j+1$) of $D(\lambda)$, while at least one $(j-1)$st minor-cofactor does not contain $(\lambda-\lambda_1)^{s+1}$; and (b), that $(\lambda-\lambda_1)^r$ is a factor of every jth minor-cofactor of $D(\lambda)$, while at least one jth minor-cofactor does not contain $(\lambda-\lambda_1)^{r+1}$, where obviously $s-r\geqq 1$. Let identities (7) be differentiated $s-1$ times successively with respect to λ, setting $\lambda=\lambda_1$, in each of the s sets of relations thus obtained. By hypothesis (a), the right members of the resulting s sets of relations vanish and, by (b), each term of the left members of the first r sets of equations also vanishes. There remain, however, the following $s-r$ non-trivial sets of relations:

$$(8)\qquad (a_{ii}-\lambda_1)\delta_{ij}{}^{(r)}+\sum_{k=1}^{n} a_{ik}\delta_{kj}{}^{(r)} = r\delta_{ij}{}^{(r-1)} = 0 \ (i=1,\cdots,n;\ k\neq i),$$

$$(9)\qquad (a_{ii}-\lambda_1)\delta_{ij}{}^{(r+1)}+\sum_{k=1}^{n}a_{ik}\delta_{kj}{}^{(r+1)}=(r+1)\delta_{ij}{}^{(r)} \qquad (i=1,\cdots,n;\ k\neq i),$$

. . . .

$$(10\quad (a_{ii}-\lambda_1)\delta_{ij}{}^{(r+h-1)}+\sum_{k=1}^{n}a_{ik}\delta_{kj}{}^{(r+h-1)}=(r+h-1)\delta_{kj}{}^{(r+h-2)} \qquad (i=1,\cdots,n;\ k\neq i),$$

. . . .

$$(11)\qquad (a_{ii}-\lambda_1)\delta_{ij}{}^{(s-1)}+\sum_{k=1}^{n}a_{ik}\delta_{kj}{}^{(s-1)}=(s-1)\delta_{ij}{}^{(s-2)} \qquad (i=1,\cdots,n;\ k\neq i).$$

It is to be remembered that at least $j-1$ of the $\delta_{1j},\cdots,\delta_{nj}$ are identically zero in λ, the remaining $n-j+1$ of the δ_{ij} being the jth minor-cofactors of the elements of some row of a $(j-1)$st minor-cofactor of $D(\lambda_1)$. Furthermore, the choice of the $(j-1)$st and jth minor-cofactors occurring in (7) is restricted only by conditions (a) and (b), where it is to be understood that at least one of the $\delta_{i,j-1}(\lambda)$ does not contain $(\lambda-\lambda_1)^{s+1}$, and at least one of the $\delta_{kj}(\lambda)$ $(k=1,\cdots,n)$ does not contain $(\lambda-\lambda_1)^{r+1}$.

If now the arbitrary ω in (2) be chosen so that $\omega=s-r-1$, it is observed that the $\omega+1$ sets of relations (8), $\cdots$, (11) have the same coefficients in their left members as the $\omega+1$ sets of equations (3), $\cdots$, (6). The $A_1{}^{(h)},\cdots,A_n{}^{(h)}$ in the left members of (5) occur in the right members of the $(h+1)$st set; corresponding relations hold in equations (8), $\cdots$, (11). One thus obtains, by comparing the sufficient conditions (3), $\cdots$, (6) with the equations (8), $\cdots$, (11), the following solutions of equations (3), $\cdots$, (6):

$$(12)\qquad A_k{}^{(1)}=C_{1j}\delta_{kj}{}^{(r)} \qquad (k=1,\cdots,n),$$

$$(13)\qquad A_k{}^{(2)}=C_{1j}\,\frac{\omega}{r+1}\,\delta_{kj}{}^{(r+1)}+C_{2j}\delta_{kj}{}^{(r)} \qquad (k=1,\cdots,n),$$

. . . .

$$(14)\quad \left\{\begin{aligned} A_k{}^{(h)}=C_{1j}\,&\frac{\omega(\omega-1)\cdots(\omega-h+2)}{(r+1)(r+2)\cdots(r+h-1)}\delta_{kj}{}^{(r+h-1)}\\ &+C_{2j}\,\frac{(\omega-1)\cdots(\omega-h+2)}{(r+1)\cdots(r+h-2)}\delta_{kj}{}^{(r+h-2)}\\ +\cdots+C_{h-1,j}\,&\frac{\omega-h+2}{r+1}\delta_{kj}{}^{(r+1)}+C_{hj}\delta_{kj}{}^{(r)} \qquad (k=1,\cdots,n),\end{aligned}\right.$$

. . . .

$$
(15)\quad \left\{\begin{aligned}
A_k^{(\omega+1)} = C_{1j}\, &\frac{\omega!}{(r+1)(r+2)\cdots(s-1)}\,\delta_{kj}^{(s-1)} \\
&+ C_{2j}\,\frac{(\omega-1)!}{(r+1)\cdots(s-2)}\,\delta_{kj}^{(s-2)} + \cdots \\
\cdots + C_{\omega+1,j}\, &\frac{2!}{(r+1)(r+2)}\,\delta_{kj}^{(r+2)} + C_{\omega j}\,\frac{1}{r+1}\,\delta_{kj}^{(r+1)} \\
&(k = 1, \cdots, n).
\end{aligned}\right.
$$

On substituting the values of the $A_k^{(i)}$ given by (12), $\cdots$, (15) in (2), ω particular solutions of system (1) are found to be

$$
(16)\quad \left\{\begin{aligned}
x_k = {} & C_{1j}e^{\lambda_1 t}\Big[\delta_{kj}^{(r)}t^{\omega} + \frac{\omega}{r+1}\,\delta_{kj}^{(r+1)}t^{\omega-1} + \frac{\omega(\omega-1)}{(r+1)(r+2)}\,\delta_{kj}^{(r+2)}t^{\omega-2} + \\
& \qquad\cdots + \frac{\omega!}{(r+1)\cdots(s-1)}\,\delta_{kj}^{(s-1)}\Big] \\
& + C_{2j}e^{\lambda_1 t}\Big[\delta_{kj}^{(r)}t^{\omega-1} + \frac{\omega-1}{r+1}\,\delta_{kj}^{(r+1)}t^{\omega-2} + \cdots + \frac{(\omega-1)!}{(r+1)\cdots(s-2)} \\
& \qquad \delta_{kj}^{(s-2)}\Big] \\
& + \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
& \quad + C_{\omega-1,j}e^{\lambda_1 t}\Big[\delta_{kj}^{(r)}t^{2} + \frac{2}{r+1}\,\delta_{kj}^{(r+1)}t + \frac{2}{(r+1)(r+2)}\,\delta_{kj}^{(r+2)}\Big] \\
& \quad + C_{\omega j}e^{\lambda_1 t}\Big[\delta_{kj}^{(r)}t + \frac{1}{r+1}\,\delta_{kj}^{(r+1)}\Big] \quad (k = 1, \cdots, n).
\end{aligned}\right.
$$

After replacing the $C_{ij}(s-i)!(\omega-i+1)!$ by $(r+1)\cdots(s-i)c_{ij}$ $(i=1, \cdots, \omega)$, the solutions (16) can be written in the form

$$
(17)\quad \left\{\begin{aligned}
x_k = {} & c_{1j}e^{\lambda_1 t}\left[\frac{\delta_{kj}^{(s-1)}}{(s-1)!} + \frac{\delta_{kj}^{(s-2)}t}{1!(s-2)!} + \cdots + \frac{\delta_{kj}^{(s-\omega-1)}t^{\omega}}{\omega!(s-\omega-1)!}\right] \\
& + c_{2j}e^{\lambda_1 t}\left[\frac{\delta_{kj}^{(s-2)}}{(s-2)!} + \frac{\delta_{kj}^{(s-3)}t}{1!(s-3)!} + \cdots + \frac{\delta_{kj}^{(s-\omega-1)}t^{\omega-1}}{(\omega-1)!(s-\omega-1)!}\right] \\
& + \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
& \quad + c_{\omega j}e^{\lambda_1 t}\left[\frac{\delta_{kj}^{(s-\omega)}}{(s-\omega)!} + \frac{\delta_{kj}^{(s-\omega-1)}t}{(s-\omega-1)!}\right] \qquad (k = 1, \cdots, n).
\end{aligned}\right.
$$

The ω particular solutions of (1) given by (17) can now be written in the convenient form

$$
(18)\qquad x_k = \sum_{\sigma=1}^{\omega} c_{\sigma j}e^{\lambda_1 t}\sum_{i=\sigma}^{\omega+1}\frac{\delta_{kj}^{(s-i)}t^{i-\sigma}}{(s-i)!(i-\sigma)!} \qquad (k = 1, \cdots, n).
$$

Thus it has been shown that if, for any root $\lambda = \lambda_1$, of $D(\lambda) = 0$ the $(j - 1)$st and jth minor cofactors of $D(\lambda)$ have properties (a) and (b); then $\omega = s - r - 1$ solutions of system (1) given by (17) or (18) can be written at once in terms of the jth minor cofactors of $D(\lambda_1)$ and their λ-derivatives. It remains to be shown that for every multiple root of $D(\lambda) = 0$ the corresponding solutions of system (1), not of the type $x_k = A_k e^{\lambda_1 t}$ $(k = 1, \cdots, n)$, can always be obtained directly from form (17) or (18).

6. *Solutions of System* (1) *Corresponding to a Multiple Root of* $D(\lambda) = 0$. Suppose (A) that $\lambda = \lambda_1$ is a root of $D(\lambda) = 0$ of order S_0, and (B) that all 1st, 2nd, $\cdots$, $(q - 1)$st minors of $D(\lambda_1)$ are zero, $(q \leqq S_0)$, while at least one qth minor of $D(\lambda_1)$ is different from zero. In consequence of (A) and (B), q purely exponential solutions of (1) are readily obtained in terms of qth minor cofactors of $D(\lambda_1)$. For, consider again the system of differential equations

$$x'_i = a_{i1} x_1 + \cdots + a_{in} x_n \qquad (i = 1, \cdots, n), \tag{1}$$

for which, if possible, solutions of the form

$$x_i = A_i e^{\lambda_1 t} \qquad (i \quad 1, \cdots, n), \tag{2}$$

are to be found. On substituting (2) in (1) there results as sufficient conditions that (2) shall be a solution of (1)

$$(a_{ii} - \lambda_1) A_i + \sum_{k=1}^{n} a_{ik} A_k = 0 \qquad (i = 1, \cdots, n;\ k \neq i), \tag{3}$$

In view of hypothesis (A) and (B) the general solution of equations (3) can be written in the form

$$A_i = c_1 \delta_{iq1} + c_2 \delta_{iq2} + \cdots + c_q \delta_{iqq} \qquad (i = 1, \cdots, n), \tag{4}$$

where the δ_{iqk} are the qth minor cofactors of $D(\lambda_1)$ obtained in solving (3) by determinants, and the c_j are arbitrary constants. At least $q - 1$ of the A_i in (4) will have single-term values of the form $A_i = c_k \delta_{iqk}$. Substitute the A_i given by (4) in (2) and one obtains the q purely exponential solutions of system (1)

$$x_k = c_1 \delta_{kq1} e^{\lambda_1 t} + c_2 \delta_{kq2} e^{\lambda_1 t} + \cdots + c_q \delta_{kqq} e^{\lambda_1 t} \qquad (k = 1, \cdots, n). \tag{5}$$

In consequence of assumption (B) all derivatives with respect to λ, up to and including the $(q - j - 1)$st, of all jth minor cofactors of $D(\lambda)$ vanish when $\lambda = \lambda_1$, for $j = 1, \cdots, (q-1)$. This follows from the fact that the λ-derivative of every $(j-1)$st minor of $D(\lambda)$ is a sum of jth minors of $D(\lambda)$. Thus hypotheses (B) imply that $(\lambda - \lambda_1)^{q-j}$ is a factor of every jth minor of $D(\lambda)$ $[j = 1, 2, \cdots, (q-1)]$. If there exists at least one first-minor of $D(\lambda)$ which does not contain $(\lambda - \lambda_1)^q$ as a factor, then the remaining $S_0 - q$ solutions of system (1), associated with $\lambda = \lambda_1$; can be written at once by (17) or (18) of Section 5, for the hypotheses of Section 5 are satisfied if

$$S = S_0, \quad \omega = S_0 - q, \quad r = q - 1, \quad j = 1.$$

It may happen, however, that $(\lambda - \lambda_1)^\rho$, where $\rho \geqq q$, is a common factor of all first minors of $D(\lambda)$ in which case there are fewer than $S_0 - q$ solutions of the form (17), of Section 5, which are expressed in terms of the first minor cofactors and their λ-derivatives. In fact, a similar special situation may exist for the jth minors of $D(\lambda)$ for $j = 2, \cdots, (q-1)$. The assumptions with respect to this question will therefore be generalized so as to include every possible special case.

Hence let it be assumed (C) that every jth minor of $D(\lambda)$ has the factor $(\lambda - \lambda_1)^{S_j}$, while at least one jth minor does not contain $(\lambda - \lambda_1)^{S_j+1}$ $(j = 1, 2, \cdots, q)$, where obviously $S_j \geqq q - j$ and $S_{j-1} - S_j \geqq 1$. Hypotheses (C) are identical in character with the assumptions underlying the derivation of forms (17) and (18), of Section 5, and consequently the results of the preceding article are immediately applicable. The solutions of equations (1) thus obtained by successive application of forms (17) and (18) of Section 5, are, in general, expressed in terms of 1st, 2nd, $\cdots$, qth minor cofactors of $D(\lambda_1)$ and their λ derivatives. In obtaining solutions in terms of jth minor-cofactors of $D(\lambda)$, $(j = 1, \cdots, q)$, one makes the following substitutions in (17) or (18), of Section 5:

$$(6) \qquad s = S_{j-1}, \quad \omega = S_{j-1} - S_j - 1 = \omega_j, \quad r = S_j.$$

The total number of solutions, corresponding to $\lambda = \lambda_1$ which one obtains in this way, is clearly given by

$$\omega_1 + \omega_2 + \cdots + \omega_q = (S_0 - S_1 - 1) + (S_1 - S_2 - 1) + \cdots \\ + (S_{q-1} - S_q - 1) = S_0 - S_q - q = S_0 - q,$$

since, by (B), $S_q = 0$. There are also q solutions depending on the root $\lambda = \lambda_1$, which are given by equations (5). Therefore there are S_0 solutions of System (1) associated with $\lambda = \lambda_1$ which is a root of $D(\lambda) = 0$ of multiplicity S_0. They are

$$(7)\quad x_k = \sum_{j=1}^{q} \sum_{\sigma=1}^{\omega} c_{\sigma j} e^{\lambda_1 t} \sum_{i=\sigma}^{\omega+1} \frac{\delta_{kj}{}^{(s-i)} t^{i-\sigma}}{(s-i)\,!\,(i-\sigma)\,!} + \sum_{i=1}^{q} c_i \delta_{kqi} e^{\lambda_1 t} \qquad (k = 1, \cdots, n).$$

The solutions corresponding to every other root of $D(\lambda) = 0$ can be found similarly.

7. *Independence of the n Solutions of System* (1). The independence of the n solutions of (1) will be established by proving that their determinant $\Delta(t)$ is distinct from zero. In order to attain complete generality in the following proof, assume that the characteristic equations $D(\lambda) = 0$ has β roots, $\lambda_1, \lambda_2, \cdots, \lambda_\beta$ of orders $e_1, e_2, \cdots, e_\beta$, respectively, where clearly $\beta \leqq n$ and $e_1 + e_2 + \cdots + e_\beta = n$. Let

$$(2)\qquad x_1 = \phi_{1i}, \cdots, x_n = \phi_{ni} \qquad (i = 1, \cdots, n),$$

represent the n solutions of the system

$$(1)\qquad x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad (i = 1, \cdots, n).$$

The determinant of the n solutions is

$$(3)\qquad \Delta(t) \equiv |\,\phi_{ij}\,| = \Delta(0)e^{(a_{11}+\cdots+a_{nn})t}.$$

The indirect method of proof will be employed. Assume

$$(4)\qquad \Delta(t) = 0,$$

which implies the existence of n constants $c_1{}^{(0)}, \cdots, c_n{}^{(0)}$, not all of which are zero, such that

$$(5)\qquad c_1{}^{(0)}\phi_{i1}(0) + \cdots + c_n{}^{(0)}\phi_{in}(0) = 0 \qquad (i = 1, \cdots, n).$$

From (5) it is evident that

$$(6)\qquad x_i = c_1{}^{(0)}\phi_{i1} + \cdots + c_n{}^{(0)}\phi_{in} \qquad (i = 1, \cdots, n),$$

is a solution of system (1) having as initial conditions

$$(7)\qquad x_1(0) = x_2(0) = \cdots = x_n(0) = 0.$$

But $x_1 \equiv x_2 \equiv \cdots \equiv x_n \underset{t}{\equiv} 0$ is also a solution of (1) having initial conditions (7). In view of the uniqueness of solutions (2) under given initial conditions, (6) becomes

$$(8) \qquad x_i = c_1^{(0)}\phi_{i1} + \cdots + c_n^{(0)}\phi_{in} \underset{t}{\equiv} 0 \qquad (i = 1, \cdots, n).$$

Any x_j of (8) can be written in the form

$$(9) \quad x_j = \sum_{i=0}^{n_1-1} c_i^{(1)} f_{1i}(t) e^{\lambda_1 t} + \cdots + \sum_{i=0}^{n_\beta-1} c_i^{(\beta)} f_{\beta i}(t) e^{\lambda_\beta t} \underset{t}{\equiv} 0,$$

where the $f_{1i}(t), \cdots, f_{\beta i}(t)$ are polynomials in t whose respective degrees are indicated by their second subscripts, while the coefficients occurring in these polynomials are either derivatives of cofactors in $D(\lambda_i)$ or linear combinations of such derivatives. The $n_1 - 1$, $n_2 - 1$, $\cdots$, $n_\beta - 1$ are the highest degrees of the β sets of polynomials $f_{1i}(t)$, $f_{2i}(t), \cdots, f_{\beta i}(t)$ respectively.

Set $n_1 + n_2 + \cdots + n_\beta = p$ and differentiate (9) $p - 1$ times with respect to t, setting $t = 0$ in each of the resulting p relations. One thus obtains the p linear homogeneous relations among the p constants $c_0^{(1)}, \cdots,$ $c^{(1)}_{n_1-1}, \cdots, c_0^{(\beta)}, \cdots, c^{(\beta)}_{n_\beta-1}$,

$$(10) \qquad x_j(0) = 0, \quad x'_j(0) = 0, \cdots, x_j^{(p-1)}(0) = 0.$$

Since (10) are consistent the determinant of the coefficients of the $c_i^{(k)}$ is zero, or

$$(11) \qquad \Gamma = \prod_{i=1}^{p} \eta_{ji}\, \Gamma_1(\lambda_1, \cdots, \lambda_\beta) = 0,$$

where η_{ji} is the factor common to the elements of the ith column in Γ, and is either a single term of the form $\delta^{(a_i)}_{j\mu_i\nu_i}(\lambda_i)$ or a linear combination of terms of that kind. In accordance with the foregoing notation $\delta^{(a_i)}_{j\mu_i\nu_i}(\lambda_i)$ indicates the a_ith λ-derivative for $\lambda = \lambda_1$ of a μ_ith minor-cofactor in $D(\lambda)$. After removing the factors $\eta_{j1}, \cdots, \eta_{jp}$ from Γ, there remains the determinant

$$(12) \quad \Gamma_1 \equiv \begin{vmatrix} (1)^{(n_1-1)}, & \cdots, & (1)^{(1)}, & 1, & \cdots, & (1)^{(n_\beta-1)}, & \cdots, & (1)^{(1)}, & 1 \\ (\lambda_1)^{(n_1-1)}, & \cdots, & (\lambda_1)^{(1)}, & \lambda_1, & \cdots, & (\lambda_\beta)^{(n_\beta-1)}, & \cdots, & (\lambda_\beta)^{(1)}, & \lambda_\beta \\ (\lambda_1^2)^{(n_1-1)}, & \cdots, & (\lambda_1^2)^{(1)}, & \lambda_1^2, & \cdots, & (\lambda_\beta^2)^{(n_\beta-1)}, & \cdots, & (\lambda_\beta^2)^{(1)}, & \lambda_\beta^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (\lambda_1^{p-1})^{(n_1-1)}, & \cdots, & (\lambda_1^{p-1})^{(1)}, & \lambda_1^{p-1}, & \cdots, & (\lambda_\beta^{p-1})^{(n_\beta-1)}, & \cdots, & (\lambda_\beta^{p-1})^{(1)}, & \lambda_\beta^{p-1} \end{vmatrix},$$

where for convenience in writing

$$(\lambda_i^j)^{(k)} = \left[\frac{d^k\lambda^j}{d\lambda^k}\right]_{\lambda=\lambda_i}.$$

In order to show that Γ_1 is not zero, replace the elements of the first column of Γ_1 by $1, \lambda, \lambda^2, \cdots, \lambda^{p-1}$ respectively, and designate the resulting determinant by $\Lambda(\lambda)$. The equation $\Lambda(\lambda) = 0$ is of degree $p - 1$ and has the roots $\lambda_1, \lambda_2, \cdots, \lambda_\beta$ of orders $n_1 - 1, n_2, n_3, \cdots, n_\beta$ respectively. Since $n_1 - 1 + n_2 + \cdots + n_\beta = p - 1$, all of the roots of $\Lambda(\lambda) = 0$ are accounted for. It is thus evident that

$$\left[\frac{d^{n_1-1}\Lambda(\lambda)}{d\lambda^{n_1-1}}\right]_{\lambda=\lambda_1} = \Gamma_1 \neq 0,$$

and consequently some η_{ji} in (11), say η_{j1}, is zero. On omitting the last equation in (10) and suppressing the terms of the first column of Γ, since $\eta_{j1} = 0$, it is seen by a repetition of the foregoing argument that a second η_{ji}, say η_{j2}, must also be zero. By this procedure, one ultimately proves that

$$(13) \qquad \eta_{j1} = \eta_{j2} = \cdots = \eta_{jp} = 0.$$

since x_j was any one of the variables $x_1, \cdots, x_n$, it is thus evident that under the assumption $\Delta(t) = 0$ one has (13) for $j = 1, \cdots, n$.

In Section 6 it was seen that for any $\lambda = \lambda_k$ the corresponding solutions of system (1) in terms of jth minor cofactors of $D(\lambda_k)$ have zero values for at least $j - 1$ of the variables $x_1, \cdots, x_n$. In view of this property it follows that for each λ_i there is an x_j for which at least one of the corresponding η_{ji} of (11) is a non-zero term of the form $\delta^{(a_i)}_{j\mu_i\nu_i}(\lambda_i)$. Hence (13) contradicts the established properties of the solutions of system (1) and therefore the assumption $\Delta(t) = 0$, on which the derivation of (13) was based, is erroneous. This establishes the fact that the n solutions of equations (1) given in Section 6 form a fundamental set.

8. *Systems of Non-Homogeneous Differential Equations.* Consider the system of non-homogeneous equations

$$(1') \qquad x'_i = a_{i1}x_1 + \cdots + a_{in}x_n + \theta_i \quad (i = 1, \cdots, n),$$

which differ from the system of homogeneous equations

$$(1) \qquad x'_i = a_{i1}x_1 + \cdots + a_{in}x_n \quad (i = 1, \cdots, n),$$

by the $\theta_i(t)$ which are assumed to be analytic functions of t. Let

$$(2) \qquad x_1 = \phi_{1i}(t), \cdots, x_n = \phi_{ni}(t) \quad (i = 1, \cdots, n),$$

represent a fundamental set of solutions of (1). Change from the dependent variables x_i to new dependent variables y_i by the transformation

$$(3)\qquad x_i = \phi_{i1}y_1 + \cdots + \phi_{in}y_n \qquad (i = 1, \cdots, n).$$

Since the ϕ_{ij} satisfy equations (1), the result of substituting (3) in (1) is

$$(4)\qquad y'_1\phi_{i1} + \cdots + y'_n\phi_{in} = \theta_i \qquad (i = 1, \cdots, n).$$

On solving (4) for the y_i, one has

$$(5)\qquad \left\{ \begin{aligned} y'_i &= \frac{1}{\Delta(t)} \left| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{j1}, & \cdots, & \phi_{j,i-1}, & \theta_j, & \phi_{j,i+1}, & \cdots, & \phi_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right| \\ &= \frac{e^{\sum_{k=1}^{n} \lambda_k t - \lambda_i t}}{\Delta(0) e^{\sum_{k=1}^{n} \lambda_k t}} \left| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{j1}, & \cdots, & p_{j,i-1}, & \theta_j, & p_{j,i+1}, & \cdots, & p_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right| \\ &= \frac{e^{-\lambda_i t}}{\Delta(0)} \left| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{j1}, & \cdots, & p_{j,i-1}, & \theta_j, & p_{j,i-1}, & \cdots, & p_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right| \quad (i = 1, \cdots, n), \end{aligned} \right.$$

where $\qquad e^{\lambda_j t} p_{ij} \equiv \phi_{ij} \qquad (i, j = 1, \cdots, n).$

On expanding the determinants occurring in (5), one can write the y'_i in the form

$$(6)\qquad y'_i = \sum_{j=1}^{n} \theta_j P_{ji} e^{-\lambda_i t} \qquad (i = 1, \cdots, n),$$

where the $P_{ji}\Delta(0)$ is the cofactor of p_{ji} in the determinant $|\, p_{ji} \,|$. Since t is the only variable occurring in the right members of (6), the y_i are obtained by quadratures, and hence

$$(7)\qquad y_i = \sum_{j=1}^{n} \int \theta_j P_{ji} e^{-\lambda_i t} dt + c_i \qquad (i = 1, \cdots, n).$$

On substituting (7) in (3), one has as the general solution of equations (1)

$$(8)\qquad x_i = \sum_{k=1}^{n} e^{\lambda_k t} p_{ik} \left\{ \sum_{j=1}^{n} \int \theta_j P_{jk} e^{-\lambda_k t} dt \right\} + \sum_{j=1}^{n} c_j \phi_{ij} \qquad (i = 1, \cdots, n).$$

A brief analysis of the nature of the solutions (8) will be given for each of the following important cases:

CASE I. The characteristic equation $D(\lambda) = 0$ has no multiple root and (a), the θ_i are polynomials in t; (b) the θ_i are linear functions of Sin at and Cos at; and (c) the θ_i are linear in Sin at and Cosat and polynomials in t.

CASE II. The characteristics equation $D(\lambda) = 0$ has one or more multiple roots which give rise to solutions involving polynomials in t, and the θ_i have the properties (a), (b), and (c) as in Case I.

Case I(a). For this case the P_{jk} are constants and therefore the general solution (8) becomes

$$(9) \qquad x_i = \sum_{j,k=1}^{n} p_{ik} P_{jk} e^{\lambda_k t} \int \theta_j e^{-\lambda_k t} dt + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n).$$

After the indicated integrations are performed (9) can be written in the form

$$(10) \qquad x_i = \psi_i(t) + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n).$$

where the ψ_i are polynomials in t whose degrees do not exceed the highest degree of t in the θ_i.

Case I(b). The quadratures occurring in (8) are now of the form

$$\int \begin{pmatrix} \text{Sin } at \\ \text{Cos } at \end{pmatrix} e^{-\lambda_k t} dt.$$

If $a^2 + \lambda^2{}_k \neq 0$ $(k = 1, \cdots, n)$, then on performing the integrations indicated in (8) one obtains as the general solution of (1′)

$$(11) \qquad x_i = R_i\,(\text{Sin } at, \text{Cos } at) + \sum_{j=1}^{n} c_j \phi_{ij} \quad (i = 1, \cdots, n),$$

where the R_i are linear in Sin at and Cos at. But if $a^2 + \lambda^2{}_k = 0$, then for $\lambda_k = + a\sqrt{-1}$,

$$\int \begin{pmatrix} \text{Sin } at \\ \text{Cos } at \end{pmatrix} e^{-\lambda_k t} dt = \tfrac{1}{2} \begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix} \left[t \pm \frac{1}{2a\sqrt{-1}} e^{-2at\sqrt{-1}} \right].$$

For $\lambda_k = - a\sqrt{-1}$, the corresponding value of the preceding integral is obtained by replacing a by $-a$. The solution of (1′) now takes the form

$$(12) \qquad x_i = R_i(\text{Sin } at, \text{Cos } at) + h_i t(\text{Cos } at \pm i \text{ Sin } at) + \sum_{j=1}^{n} c_j \phi_{ij} \qquad (i = 1, \cdots, n),$$

where the R_i are of the same form as in (11) and the h_i are constants.

Case I(c). For this case the quadratures of (8) are of the type

$$\int \begin{pmatrix} \text{Sin } at \\ \text{Cos } at \end{pmatrix} t^m e^{-\lambda_k t} dt.$$

After performing these integrations the general solution of (1′) is

$$(13) \qquad x_i = H_i(t, \text{Sin } at, \text{Cos } at) + \sum_{j=1}^{n} c_j\phi_{ij} \quad (i = 1, \cdots, n),$$

where the Sin at and Cos at enter linearly but have coefficients that are either constants or polynomials in t of the same degree as the θ_i. The special case for which $\lambda_k = \pm a\sqrt{-1}$ makes no change in the form of solution (13) except that the degree of the polynomials in t is increased by unity.

CASE II. For each of the cases (a), (b), and (c), the P_{jk} of (8) are polynomials in t. In II(a) the integrals are of the same type as those in I(a) and consequently the solution of (1′) for this case is of form (10) where, in general, the degrees of the ψ_i exceed the maximum degree of the θ_i. Cases II(b) and II(c) lead to quadratures of the same type as those of I(c) and hence the solution of (1′) in either of these two cases can be written in the form (13).

UNIVERSITY OF CHICAGO,
May 20, 1924.

On Generalizations of the Bernoullian Functions and Numbers.

By E. T. Bell.

1. Both the numbers and the functions of Bernoulli can be generalized in an infinity of ways. Many of these generalizations have either an interest of their own or an importance in analysis. The *methods* by which generalizations are attainable can conveniently be segregated into two classes. Into the first fall all methods dependent upon modifications of the difference equations defining the Bernoullian functions. The recent generalizations of Nörlund * belong to the first class. As we shall not again refer to this class it is unnecessary to describe it further. Into the second fall all methods dependent upon modifications of the symbolic equations of definition of the Bernoullian functions. The symbolic calculus implied is that of Blissard (and independently also of Lucas), in which the symbol A is the umbra, or representative, of the class of numbers or functions A_n $(n=0, 1, 2, \cdots)$, and A_n is written A^n until after the completion of all ordinary algebraic processes, when the exponents of umbrae are degraded to suffixes. During the intermediate steps powers $A^n, B^n, \cdots$ of umbrae $A, B, \cdots$ are treated as ordinary algebraic quantities, or ordinaries. For example, the first derivative of A^n with respect to A is nA^{n-1}, umbrally, and this is nA_{n-1} finally. We shall illustrate the second class of generalizations by an interesting example, which itself generalizes an extension of the Bernoullian functions and numbers due to Krause and Appell.

2. As the substance of Blissard's long neglected papers is now well known, it will be sufficient here to recall a few essentials of his calculus as we shall use it.

A function $f(x^{(1)}, x^{(2)}, \cdots, x^{(k)})$ in which each $x^{(i)}$ $(i=1, \cdots, k)$ is replaced by the umbra A is evaluated as follows: the $x^{(i)}$ are replaced by k *different* letters $A, B, \cdots, C$, each of which denotes an umbra, and *after* the degradation of exponents each of $A, B, \cdots, C$ is replaced by A. For example,

$$(A+Aa)^1 \equiv (A+Ba)^1 = A^1B^0a^0 + A^0B^1a^1 = A_1B_0 + A_0B_1a,$$

---

* "Mémoire sur les polynomes de Bernoulli," *Acta Mathematica*, 43 (1922), pp. 121-196.

if a is an ordinary, and the final value is therefore $A_0A_1(1+a)$. Note particularly that zeroth powers of umbrae must always be indicated explicitly. In particular, $A, B, \cdots, C$ being umbrae, $a, b, \cdots, c$ ordinaries,

$$(aA+bB+\cdots+cC)^0 = A_0B_0\cdots C_0;$$

and the umbral multinomial theorem is, for $n \geqq 0$ an integer (this theorem can be extended to negative and fractional exponents),

$$(aA+bB+\cdots+cC)^n = \frac{n!}{\alpha!\ \beta!\cdots\gamma!}\ a^\alpha b^\beta\cdots c^\gamma\ A_\alpha B_\beta\cdots C_\gamma,$$

the $\sum$ extending to all sets $(\alpha, \beta, \cdots, \gamma)$ of integers $\alpha, \beta, \cdots, \gamma$ such that

$$0 \leqq \alpha, \beta, \cdots, \gamma; \qquad \alpha+\beta+\cdots+\gamma = n.$$

The umbral exponential theorem is, θ being a parametric ordinary, A an umbra,

$$e^{A\theta} = \sum_{n=0}^{\infty} A^n\theta^n/n! = \sum A_n\theta^n/n! \quad (0!=1).$$

This can be considered as an identity of definition. It follows from this definition that we do *not* have, for A an *arbitrary* umbra, $e^{-A\theta}$ equal to the reciprocal of $e^{A\theta}$. The value of $e^{-A\theta}$ is obtained from that of $e^{A\theta}$ by changing the sign of the parameter θ in $e^{A\theta}$:

$$e^{-A\theta} = \sum_{n=0}^{\infty} (-1)^n A_n\theta^n/n!.$$

The one exception in which $e^{-A\theta}$ is the reciprocal of $e^{A\theta}$ is the trivial case in which $A_n = 1$ $(n=0, 1, 2, \cdots)$.

Infinite series, such as those just discussed, are used merely as generating functions, or *generators*. Thus $e^{A\theta}$ is the generator of $A_n/n!$, this being the coefficient of θ^n. The question of convergence of generators can be disregarded. If this is not obvious, it has been proved in a former paper.* In particular we can equate coefficients of like powers of θ in identities between generators derived by means of the umbral calculus.

It is desirable to have no umbral derivatives in the final statements of results. Such derivatives can always be eliminated by repeated applications of the immediately verifiable identity

$$\frac{\partial^k}{\partial A^k}(\theta+A)^n = \frac{\partial^k}{\partial\theta^k}(\theta+A)^n,$$

in which θ is an ordinary and A an umbra.

---

* *Transactions of the American Mathematical Society*, Vol. 25, pp. 135-154.

Addition of umbrae $v, p, \cdots, t$ is used here only in the following sense. If u is the sum $v + p + \cdots + t$ of $v, p, \cdots, t$, this is written $u = v + p + \cdots + t$, and means that for all integers $n \geqq 0$,

$$u^n \equiv u_n = (v + p + \cdots + t)^n,$$

the last of which is to be expanded by the umbral multinomial theorem. Thus if $u = v + p$, then $u_1 = v_1 p_0 + v_0 p_1$, $u_2 = v_2 p_0 + 2 v_1 p_1 + v_0 p_2$, etc.

3. *Henceforth each of the k letters $A, B, \cdots, C$ represents an umbra of the Bernoulli numbers.*

$$D_0 = 1,\ D_1 = -\tfrac{1}{2},\ D_2 = \tfrac{1}{6},\ D_4 = -1/30, \cdots, D_{2n+1} = 0 \quad (n > 0),$$
$$D = A, B, \cdots, C,$$

and each of the k letters $X, Y, \cdots, Z$ represents an umbra of the common Bernoullian functions with the respective ordinary arguments $x, y, \cdots, z$, so that

$$nW_{n-1} \equiv nW_{n-1}(w) = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} B_r w^{n-r} \quad (n \geqq 0),$$

$$(W, w) \equiv (X, x), \quad (Y, y), \cdots, \quad (Z, z).$$

It is well known (or follows as the simplest case, $k = 1$, of the numbers Ω and functions Λ later discussed in this paper) that the symbolic equations of definition for D, W are

$$(1) \qquad \frac{\theta}{e^\theta - 1} = e^{D\theta}, \quad \frac{e^\theta (1 - e^{w\theta})}{1 - e^\theta} = e^{W\theta}.$$

The arguments $w = x, y, \cdots, z$ are not assumed to be integers unless so stated.

We note first that D_n, W_n are functions of the integer variable n. A generalization which does not seem to have been considered is that in which D, W are functions of p integer variables $\xi, \eta, \cdots, \zeta$,

$$D_{\xi, \eta, \ldots, \zeta}, \quad W_{\xi, \eta, \ldots, \zeta};$$

that is, these D, W are members of p-fold sequences. A discussion of the case $p = 1$ is a necessary preliminary for the case $p > 1$. We shall therefore attend exclusively to $p = 1$, reserving the similarly treated case $p > 1$ for another occasion.

There is a second possible generalization, applicable to $p \geqq 1$, which we shall pass over here. We have postulated that the arguments $x, y, \cdots, z$

shall be ordinaries. The case in which $x, y, \cdots, z$ are umbral, so that $x^n = x_n$, etc., is treated with equal facility by means of the theory developed in a former paper.*

A third prolific generalization, immediately applicable to all that precedes, consists in removing the restriction that F, introduced presently, shall be algebraic in its arguments, $a, b, x', y', \cdots, z'$. It is now sufficiently evident that the number of generalizations which can be systematically developed is infinite. The possibilities thus suggested will be more readily appreciated after the short treatment of the special case to which this paper is devoted.

4. A power series $\sum_0^\infty t_n\theta^n$ in θ can be written in the umbral form $e^{u\theta}$, where $u_n = n!\, t_n$. Let $F \equiv F(a, b, x', y', \cdots, z')$ be a rational algebraic function of its ordinary arguments $a, b, x', y', \cdots, z'$, and in F make the substitution

$$(a, b, x', y', \cdots, z') \equiv (\theta, e^\theta, e^{x\theta}, e^{y\theta}, \cdots, e^{z\theta}),$$

so that F becomes a function of θ, say $f(\theta)$. Write the power series expansion of $f(\theta)$ in the form $e^{u\theta}$. Then u is an umbra of a system of polynomials in the k variables $x, y, \cdots, z$, and $u^n \equiv u_n = u_n\ (x, y, \cdots, z)$ is the nth such polynomial. For the special values $(x, y, \cdots, z) \equiv (x_0, y_0, \cdots, z_0)$, where $x_0, y_0, \cdots, z_0$ are given constants, the polynomials u become a system of numbers β. We shall say that the numbers β are *associated* with the polynomials u, and that they have the *characteristic* determined by $(x_0, y_0, \cdots, z_0)$. Evidently u is uniquely known when F is given, and likewise for β when in addition the characteristic is assigned.

For simplicity let us restrict the discussion to the case in which the coefficients of F are rational, and the domain (of reducibility) is the rational domain. Then the resolution of the numerator of F into irreducible polynomials in $a, b, x', y', \cdots, z'$ is unique, and likewise for the denominator. Suppose common factors to have been removed. This unique form of F is expanded in the form $e^{u\theta}$, and u is now called a *primary* umbra. From the definitions, primary umbrae arise only from irreducible rational algebraic functions F. The numbers associated with u are called primary. Nonprimaries, either polynomials or their respective associated numbers, are called derived.

A further generalization, easily attainable and readily developed by the symbolic method, at once suggests itself. In the foregoing we have replaced $x', y', \cdots, z'$ by $e^x, e^y, \cdots, e^z$ respectively. But we might also replace

---

* *Mathematische Zeitschrift*, Bd. 19, S. 35.

$x', y', \cdots, z'$ by $e^{P(x)}, e^{Q(y)}, \cdots, e^{R(z)}$, where $P, Q, \cdots, R$ are single valued functions. The case where $P, Q, \cdots, R$ are algebraic is of special interest, because it leads to an immediate generalization of the Bessel coefficients.

It is now evident that we can set up a simple isomorphism between the multiplicative properties of rational algebraic functions in $k+2$ variables $a, b, x, y, \cdots, z$ and the additive properties of umbrae of polynomials in k variables $x, y, \cdots, z$, and similarly for umbrae of the respective associated numbers.

Suppose first that F is irreducible. Let $e^{u\theta}$ be the umbral exponential obtained from F as above outlined. Then from F^n, $n > 0$ an integer, is derived $e^{n.u\theta}$, *where $n.u$ is a convenient symbol for the sum $u+v+\cdots+t$ of n umbrae, each of which is to be replaced by u after the degradation of exponents.* Thus

$$(3.u)^2 = (u+v+t)^2 = u_2v_0t_0 + u_0v_2t_0 + u_0v_0t_2 + 2u_1v_1t_0 + 2u_1v_0t_1 + 2u_0v_1t_1.$$

If F is reducible, say $F = G^\alpha H^\beta \cdots K^\gamma$, let the respective umbral exponentials of $F, G, H, \cdots, K$ be $e^{u\theta}, e^{v\theta}, e^{t\theta}, \cdots, e^{p\theta}$, where $u, v, t, \cdots, p$ are umbrae. Then clearly since for u', v' arbitrary umbrae, $e^{u'\theta}e^{v'\theta} = e^{(u'+v')\theta}$, we have

$$u = \alpha.v + \beta.t + \cdots + \gamma.p.$$

With reference to umbrae, it follows (cf. § 6), that the rational algebraic functions giving rise to exponentials involving those umbrae play a part analogous to that of antilogarithms in ordinary multiplication. There is also a complete correspondence for division, but we shall not discuss this here.

From the above sketch we see that the algebra of derived umbrae can be inferred from that of the primaries. The common Bernoullian functions X defined in (1) are primaries, since in this case F is $F(e^\theta, e^{\theta x})$, and $F(a, b, x', y', \cdots, z') \equiv F(b, x') = b(1-x')/(1-b)$, which is irreducible in the sense defined.

5. To illustrate the foregoing we take the immediate extension suggested by $F(b, x')$, of § 4 and multiply together k such rational functions. The k letters $a, b, \cdots, c$ denote ordinaries; $x, y, \cdots, z$ are k independent variables, and as in § 3 each of the k umbrae $A, B, \cdots, C$ is an umbra of the Bernoulli numbers.

The Functions Λ,

$$\Lambda^n \equiv \Lambda_n \equiv \qquad \Lambda_n^{(a,\, b,\, \cdots,\, c)}\,(x,\, y,\, \cdots,\, z)$$

are defined by

$$(2) \qquad e^{\Lambda\theta} = \frac{e^{a\theta}(1-e^{ax\theta})}{1-e^{a\theta}} \cdot \frac{e^{b\theta}(1-e^{by\theta})}{1-e^{b\theta}} \cdots \frac{e^{c\theta}(1-e^{cz\theta})}{1-e^{c\theta}} \,.$$

With Λ it is evidently convenient to introduce the following, suggested by the right of (2): the functions of Ψ,

$$\Psi^n \equiv \Psi_n \equiv \Psi_n^{(a,\, b,\, \cdots,\, c)}\,(x,\, y,\, \cdots,\, z),$$

defined by

$$(3) \qquad e^{\Psi\theta} = (1-e^{ax\theta})(1-e^{by\theta}) \cdots (1-e^{cz\theta})\,;$$

the numbers Ω,

$$\Omega^n \equiv \Omega_n \equiv \Omega_n^{(a,\, b,\, \cdots,\, c)},$$

whose definition is

$$(4) \qquad e^{\Omega\theta} = \frac{a\theta}{e^{a\theta}-1} \cdots \frac{b\theta}{e^{b\theta}-1} \cdots \frac{c\theta}{e^{c\theta}-1}\,;$$

the numbers Θ,

$$\Theta^n \equiv \Theta_n \equiv \Theta_n^{(a,\, b,\, \cdots,\, c)},$$

defined by

$$(5) \qquad e^{\Theta\theta} = (1-e^{a\theta})(1-e^{b\theta}) \cdots (1-e^{c\theta}),$$

and the functions Φ,

$$\Phi^n \equiv \Phi_n \equiv \Phi_n^{(a,\, b,\, \cdots,\, c)}\,(x,\, y,\, \cdots,\, z),$$

whose definition is

$$(6) \qquad e^{\Phi\theta} = \frac{ax\theta}{e^{ax\theta}-1} \cdot \frac{by\theta}{e^{by\theta}-1} \cdots \frac{cz\theta}{e^{cz\theta}-1}\,.$$

Each of Λ_n, Ψ_n, Ω_n, Θ_n, Φ_n, is of *rank* n, *order* k (= the number of independent ordinary variables $x, y, \cdots, z$, or the number of ordinary parameters $a, b, \cdots, c$), and the one-rowed matrix $(a, b, \cdots, c)$ is called the *characteristic* in each case; $x, y, \cdots, z$ are the arguments of the functions.

For the characteristics

$$(1, 0, 0, \cdots, 0), \quad (0, 1, 0, \cdots, 0), \cdots, \quad (0, 0, 0, \cdots, 1),$$

Λ_n is equal to X_n, Y_n, $\cdots$, Z_n respectively, so that the common Bernoullian functions are included in the special case $k = 1$. For the characteristics

$$(a, b, 0, \cdots, 0), \quad (a, 0, c, \cdots, 0), \cdots$$

we have $\frac{1}{2}k(k-1)$ functions of order 2, of which a typical one of rank n is

$$\sum \frac{n!}{\xi!\,\eta!} a^\xi b^\eta X_\xi Y_\eta, \quad 0 \leqq \xi, \eta; \quad \xi + \eta = n,$$

so that the functions of Krause and Appell * are given by the special case $k = 2$ with the further specialization that x, y be integers $\geqq 0$. If $x, y, \cdots, z$ are integers, the explicit form of Λ_n is obtained at once from (2) rewritten as

$$e^{\Lambda\theta} = \sum_{\xi=1}^{x} \sum_{\eta=1}^{y} \cdots \sum_{\zeta=1}^{z} e^{(\xi a + \eta b + \cdots + \zeta c)\theta};$$

so that

$$\Lambda_n = \sum_{\xi=1}^{x} \sum_{\eta=1}^{y} \cdots \sum_{\zeta=1}^{z} (\xi a + \eta b + \cdots + \zeta c)^n.$$

The like is not true, of course, if $x, y, \cdots, z$ are not integers > 0.

From (4), (6) we have

$$\Phi_n^{(a, b, \cdots, c)}(1, 1, \cdots, 1) = \Omega_n^{(a, b, \cdots, c)}; \tag{7}$$

and from (3), (5),

$$\Psi_n^{(a, b, \cdots, c)}(1, 1, \cdots, 1) = \Theta_n^{(a, b, \cdots, c)}. \tag{8}$$

For the respective characteristics

$$(a, 1, 1, \cdots, 1), \quad (1, b, 1, \cdots, 1), \cdots, \quad (1, 1, 1, \cdots, c),$$

Ω_n becomes $a^n A_n$, $b^n B_n, \cdots, c^n C_n$, where as always each of $A, B, \cdots, C$ is an umbra of the Bernoulli numbers.

6. The relations between the numbers and functions defined in § 5 for various ranks are obtained by the following entirely general symbolic method. Let θ denote a parametric ordinary, and δ, $\alpha, \beta, \cdots, \gamma$ umbrae such that

$$\delta = \alpha + \beta + \cdots + \gamma, \tag{9}$$

or what is the same by the definition of umbral addition,

$$\delta^n = (\alpha + \beta + \cdots + \gamma)^n \quad (n = 0, 1, 2, \cdots). \tag{10}$$

---

* *Ber. Ges. Leipzig*, 55 (1903), pp. 39-62; *Archiv Math. Phys.* (3) 4 (1902), pp. 292-5.

Let the c_n $(n = 0, 1, 2, \cdots)$ denote ordinaries, and write

$$f(\theta) \equiv \sum_{n=0}^{r} c_n \theta^n, \tag{11}$$

where the upper limit r of the summation may be finite or infinity. If the latter, we shall assume for brevity that all infinite series occurring henceforth are absolutely convergent for some $|\theta| \neq 0$. This assumption is not a necessity.* From (11) we have by means of (10)

$$f(\theta + \delta) = \sum_{n=0}^{r} c_n \left[\sum_{s=0}^{n} \binom{n}{s} \theta^{n-s} (\alpha + \beta + \cdots + \gamma)^s \right],$$

$$= \sum_{n=0}^{r} c_n (\theta + \alpha + \beta + \cdots + \gamma)^n,$$

and therefore

$$f(\theta + \delta) = f(\theta + \alpha + \beta + \cdots + \gamma); \tag{12}$$

that is, from (9) we infer (12), where $f(\theta)$ is arbitrary except that is possesses a power series expansion (which may terminate) in θ.

To obtain umbral identities such as (9), we observe that, if $P(\theta)$, $Q(\theta)$, $\cdots$, $T(\theta)$ are ordinary functions of θ whose power series expansions in θ are given by the umbral equivalents $e^{\delta\theta}$, $e^{\alpha\theta}$, $e^{\beta\theta}$, $\cdots$, $e^{\gamma\theta}$ respectively, so that we have the symbolic equations

$$P(\theta) = e^{\delta\theta}, \quad Q(\theta) = e^{\alpha\theta}, \quad R(\theta) = e^{\beta\theta}, \cdots, \quad T(\theta) = e^{\gamma\theta},$$

and if moreover we have the ordinary identity

$$P(\theta) = Q(\theta)\, R(\theta) \cdots T(\theta), \tag{13}$$

then

$$e^{\delta\theta} = e^{(\alpha+\beta+\cdots+\gamma)\theta}, \tag{14}$$

and hence we have (9). Hence the deduction of additive umbral identities such as (9) is reduced to that of multiplicative ordinary identities.

Consider next the frequently occurring modification of (13):

$$K\theta^r P(\theta) = Q(\theta)\, R(\theta) \cdots T(\theta), \tag{15}$$

where K is an ordinary constant (independent of θ) and r an integer > 0. Then, instead of (14) we have

$$K\theta^r e^{\delta\theta} = e^{(\alpha+\beta+\cdots+\gamma)\theta},$$

---

* In the paper cited, I have shown how the contrary case is to be interpreted.

which can be written symbolically,

$$K \frac{\partial^r e^{\delta\theta}}{\partial \delta^r} = e^{(\alpha+\beta+\cdots+\gamma)\theta},$$

and hence

$$K \frac{\partial^r \delta^n}{\partial \delta^r} = (\alpha+\beta+\cdots+\theta)^n;$$

whence, as in deriving (12), and eliminating the umbral derivative by means of

$$\frac{\partial^r}{\partial \delta^r}(\theta+\delta)^n = \frac{\partial^r}{\partial \theta^r}(\theta+\delta)^n,$$

we get, as the relation implied by (16),

$$(16) \qquad K \frac{\partial^r}{\partial \theta^r} f(\theta+\delta) = f(\theta+\alpha+\beta+\cdots+\gamma).$$

Hence by means of the simplest multiplicative identities between the ordinary functions defining umbrae we pass immediately to the generalized recurrences (12), (16). The process can evidently be applied as above to any functions $\delta_n, \alpha_n, \beta_n, \cdots, \gamma_n$ $(n = 0, 1, 2, \cdots)$ which are coefficients in power series expansions.* The coefficients themselves may be, as above, functions of several variables.

7. By means of the processes in § 6 the relations between the functions defined in § 5 can be obtained with great ease. Change θ into $-\theta$ in (2) and compare the result with (2). Then, for $n = 0, 1, 2, \cdots$,

$$[a(x+1)+b(y+1)+\cdots+c(z+1)-\Lambda]^n = \Lambda^n,$$

and therefore

$$[-a(x+1)-b(y+1)-\cdots-c(z+1)+\Lambda]^n = (-\Lambda)^n,$$

from which, by (12), it follows that

$$(17) \quad \begin{aligned} f(\theta-\Lambda+a(x+1)+b(y+1)+\cdots+c(z+1)) &= f(\theta+\Lambda), \\ f(\theta+\Lambda-a(x+1)-b(y+1)-\cdots-c(z+1)) &= f(\theta-\Lambda). \end{aligned}$$

The special case in which Λ_n degenerates to the ordinary Bernoullian function X_n is obtained from (17) by taking $a = 1$ and deleting the terms in $b, \cdots, c$:

---

* If, for example, $R(\theta) = \sum \rho_n \theta^n$ defines ρ_n, we first replace ρ_n by $n!\rho'_n$, so that

$$R(\theta) = \sum \rho'_n \theta^n/n! = e^{\rho'\theta},$$

and in the final results write $\rho'_n \equiv \rho_n/n!$

$$f(\theta \mp X \pm x \pm 1) = f(\theta \pm X), \tag{18}$$

the upper or the lower signs, as always, being taken throughout. For the functions of Krause and Appell only the a, b terms are retained from those in $a, b, \cdot \cdot \cdot, c$, and further, x, y are restricted to be integers. Likewise in all subsequent formulas concerning Λ or the functions Ψ, Φ.

In exactly the same way (3) gives

$$f(\theta \pm \Psi) = (-1)^k f(\theta \mp \Psi \pm ax \pm by \pm \cdot \cdot \cdot \pm cz), \tag{19}$$

and (6) gives

$$f(\theta \pm \Phi) = f(\theta \mp \Phi \mp ax \mp by \mp \cdot \cdot \cdot \mp cz). \tag{20}$$

Proceeding similarly with (4), (5) we get for the numbers Ω, Θ the following,

$$f(\theta \pm \Omega) = f(\theta \mp \Omega \mp a \mp b \mp \cdot \cdot \cdot \mp c), \tag{21}$$

which could have been written down from (20) by taking in the latter $(x, y, \cdot \cdot \cdot, z) = (1, 1, \cdot \cdot \cdot, 1)$; and

$$f(\theta \pm \Theta) = (-1)^k f(\theta \mp \Theta \pm a \pm b \pm \cdot \cdot \cdot \pm c), \tag{22}$$

which follows also from (19).

Corresponding to the relations between the common Bernoullian functions X_n and the Bernoulli numbers B_n for various ranks n, we have here the like for Λ_n, Ω_n, and the latter include the former as special cases. For (2), (4) with θ replaced by $-\theta$, (3) and (6) give immediately

$$(-1)^k ab \cdot \cdot \cdot c \; \frac{\partial^k}{\partial \theta^k} f(\theta \pm \Lambda) = f(\theta \pm \Psi \mp \Omega). \tag{23}$$

Further general relations for Λ are obtained from (2), (5),

$$f(\theta + \Lambda + \Theta - a - b - \cdot \cdot \cdot - c) = f(\theta + \Psi); \tag{24}$$

from (3), (5), and (2) with θ replaced by $-\theta$,

$$f(\theta \mp \Lambda \pm \Theta) = f(\theta \pm \Psi \mp ax \mp by \mp \cdot \cdot \cdot \mp cz); \tag{25}$$

from (2), (6),

$$f(\theta + \Lambda + \Phi) = xy \cdot \cdot \cdot zf(\theta + \Omega + a + b + \cdot \cdot \cdot + c), \tag{26}$$

and from (2), (6), the latter with θ replaced by $-\theta$,

$$(27)\qquad f(\theta+\Lambda-\Phi)$$
$$=xy\cdots zf(\theta+\Omega+a(1+x)+b(1+y)+\cdots+c(1+z)).$$

From (4), (5) we have

$$e^{(\Omega+\Theta)\theta}=(-1)^k ab\cdots c\theta^k, =(-1)^k ab\cdots ce^{\epsilon\theta},$$

where $\epsilon_k = k\,!$, $\epsilon_n = 0$, $n \neq k$, and therefore

$$(28)\qquad f(\theta+\Omega+\Theta)=(-1)^k ab\cdots cf(\theta+\epsilon).$$

As an example of another type which, however, can be deduced from the foregoing by properly choosing f, let $\sum_2$ denote a summation with respect to all sets of 2 chosen from $ax, by, \cdots, cz$, a typical set of 2 being ax, by, and similarly for $\sum_r$, referring to all sets of r. Then from (2), and (4) with θ replaced by $-\theta$, we have

$$(-1)^k ab\cdots c\ \frac{\partial^k}{\partial\Lambda^k}\ e^{\Lambda\theta}=e^{-\Omega\theta}\ (1-e^{ax\theta})(1-e^{by\theta})\cdots(1-e^{cz\theta}),$$
$$=e^{-\Omega\theta}-\textstyle\sum_1 e^{(-\Omega+ax)\theta}+\sum_2 e^{(-\Omega+ax+by)\theta}-\cdots+(-1)^k e^{(-\Omega+ax+\cdots+cz)\theta}\ ;$$

whence, as in deducing (12), (16), we get

$$(29)\qquad (-1)^k ab\cdots c\ \frac{\partial^k}{\partial\theta^k} f(\theta\pm\Lambda)$$
$$=f(\theta\mp\Omega)-\textstyle\sum_1 f(\theta\mp\Omega\pm ax)+\sum_2 f(\theta\mp\Omega\pm ax\pm by)-\cdots.$$

The similar theorem for Ω is found from (4) rewritten in the form

$$(-1)^k ab\cdots c\theta^k=e^{\Omega\theta}\ (1-e^{a\theta})(1-e^{b\theta})\cdots(1-e^{c\theta}),$$

$$(30)\qquad ab\cdots c\ \frac{1}{k\,!}\ \frac{\partial^k}{\partial\theta^k}\ f(\theta)$$
$$=f(\theta+\Omega)-\textstyle\sum_1 f(\theta+\Omega+a)+\sum_2 f(\theta+\Omega+a+b)-\cdots.$$

The power of the symbolic method is well seen from the derivation of general addition theorems. Let $(a', b', \cdots)$, $(x', y', \cdots)$ be the characteristic and arguments of Z', $(a'', b'', \cdots)$, $(x'', y'', \cdots)$ those of Z'', where Z', Z'' are not necessarily of the same order (same number of arguments $x', \cdots, x'', \cdots$), where Z denotes any one of Λ, Ψ, Φ. Then from (2), (3), (6) it follows that

$$e^{Z'\theta}\cdot e^{Z''\theta}=e^{Z\theta},$$

where Z is the function whose characteristic and arguments are $(a', b', \cdots, a'', b'', \cdots)$, $(x', y', \cdots, x'', y'', \cdots)$ respectively. Hence

$$f(\theta+Z)=f(\theta+Z'+Z''). \tag{31}$$

If in the foregoing we take $x'=y'=\cdots=x''=y''=\cdots=1$, when Z is either Φ or Ψ we get

$$f(\theta+T)=f(\theta+T'+T''), \tag{32}$$

where T is either Ω or Θ.

As a last illustration of the power of this method let us find the partial derivatives of Λ, Ψ, Φ with respect to any of their arguments, say x. If Z_n is a function of x, and Z'_n its derivative with respect to x, it is readily seen that the first of the following implies the second,

$$F(x,\,\theta)=e^{Z\theta},\quad \frac{\partial}{\partial x}\,F(x,\,\theta)=e^{Z'\theta}.$$

To apply this to Λ, Ψ, Φ, take the logarithmic derivatives with respect to x of (2), (3), (5) and proceed as before, getting

$$x\,\frac{\partial\Lambda}{\partial x}=(\Lambda+aAx+ax)^n, \tag{33}$$

$$x\,\frac{\partial\Psi_n}{\partial x}=(\Psi+aAx+ax)^n, \tag{34}$$

$$x\,\frac{\partial\Phi_n}{\partial x}=\Phi_n-(\Phi+aAx+ax)^n, \tag{35}$$

in which, we recall, A is an umbra of the Bernoulli numbers. Applied to (1), this gives for the common Bernoullian functions X,

$$\frac{\partial X_n}{\partial x}=(x-A)^n; \tag{36}$$

whence, integrating, we obtain

$$X_n=\frac{1}{n+1}\,(x-A)^{n+1}, \tag{37}$$

it being readily seen that the constant of integration is zero.

It may be mentioned in conclusion that the application of the symbolic method to Taylor's theorem for functions of k variables, together with the preceding relations involving functions f of the several umbrae, leads at once to interesting generalizations of several well known expansions due to Euler, Boole and others in the calculus of finite differences.

UNIVERSITY OF WASHINGTON.

The Construction of Certain Periodic Orbits of the Three Body Problem.

By H. E. Buchanan.

In a recent paper * the author proved the existence of certain periodic orbits of three finite masses near the LaGrangian circular solutions. The present paper is for the purpose of constructing those solutions and discussing the geometrical character of the orbits.

The differential equations which define the variations from the circular orbits are

$$\left.\begin{aligned}
&\frac{d^2x_1}{dt^2}-2\omega(1+\delta)\frac{dy_1}{dt}-(1+\delta)^2\{(\omega^2+2A_1)x_1+2B_1x_3\}=(1+\delta)^2\epsilon\\
&\qquad\left\{x_1^2\frac{\partial^3U}{\partial x_1^3}+x_1x_3\frac{\partial^3U}{\partial x_1^2\partial x_3}+x_3^2\frac{\partial^3U}{\partial x_1\partial x_3^2}+\sum y_ky_j\frac{\partial^3U}{\partial x_1\partial y_k\partial y_j}\right.\\
&\qquad\qquad\left.+\sum z_kz_j\frac{\partial^3U}{\partial x_1\partial z_k\partial z_j}\right\}+\cdots,\\
&\frac{d^2x_3}{dt^2}-2\omega(1+\delta)\frac{dy_3}{dt}-(1+\delta)^2\{(2B_1x_1+(\omega^2+2A_3)x_3\}=(1+\delta)^2\epsilon\\
&\qquad\left\{x_1^2\frac{\partial^3U}{\partial x_3\partial x_1^2}+x_1x_3\frac{\partial^3U}{\partial x_3^2\partial x_1}+x_3^2\frac{\partial^3U}{\partial x_3^3}+\sum y_ky_j\frac{\partial^3U}{\partial x_3\partial y_k\partial y_j}\right.\\
&\qquad\qquad\left.+\sum z_kz_j\frac{\partial^3U}{\partial x_3\partial z_k\partial z_j}\right\}+\cdots,\\
&\frac{d^2y_1}{dt^2}+2\omega(1+\delta)\frac{dx_1}{dt}-(1+\delta)^2\{(\omega^2-A_1)y_1-B_1y_3\}=(1+\delta)^2\epsilon\\
&\qquad\left\{x_1y_1\frac{\partial^3U}{\partial y_1^2\partial x_1}+x_1y_3\frac{\partial^3U}{\partial y_1\partial y_3\partial x_1}+x_3y_1\frac{\partial^3U}{\partial y_1^2\partial x_3}+x_3y_3\frac{\partial^3U}{\partial y_1\partial y_3\partial x_3}\right\}+\cdots,\\
&\frac{d^2y_3}{dt^2}+2\omega(1+\delta)\frac{dx_3}{dt}-(1+\delta)^2\{\omega^2-A_3)y_3-B_3y_1\}=(1+\delta)^2\epsilon\\
&\qquad\left\{x_1y_1\frac{\partial^3U}{\partial y_1\partial y_3\partial x_1}+x_1y_3\frac{\partial^3U}{\partial y_3^2\partial x_1}+x_3y_1\frac{\partial^3U}{\partial y_3\partial y_1\partial x_3}+x_3y_3\frac{\partial^3U}{\partial y_3^2\partial x_3}\right\}+\cdots;\\
&\frac{d^2z_1}{dt^2}+(1+\delta)^2(A_1z_1+B_1z_3)=(1+\delta)^2\epsilon\sum x_kz_j\frac{\partial^3U}{\partial z_1\partial z_j\partial x_k}+\cdots,\\
&\frac{d^2z_3}{dt^2}+(1+\delta)^2(B_3z_1+A_3z_3)+(1+\delta)^2\epsilon\sum x_kz_j\frac{\partial^3U}{\partial z_3\partial z_j\partial x_k}+\cdots,
\end{aligned}\right\}(1)$$

---

* *American Journal of Mathematics*, Vol. XLV, Apr. 1923.

where the Σ denotes a sum of the form $Ay_1^2 + By_1y_3 + Cy_3^2$ and all the constants have been defined in the paper referred to.

Orbits wth Period $2\pi/\nu$. We have demonstrated the existence of solutions of equations (1) which have the following properties:

(a) $$x_i = \sum_{j=0}^{\infty} x_{ij}\epsilon^j, \qquad y_i = \sum_{j=0}^{\infty} y_{ij}\epsilon^j, \quad z_i = \sum_{j=0}^{\infty} z_{ij}\epsilon^j, \quad (i = 1,\ 3),$$

$$\delta = \sum_{j=0}^{\infty} \delta_j\epsilon^j, \qquad \alpha_i = \sum_{j=0}^{\infty} \alpha_{ij}\epsilon^j,\ \beta_i = \sum \beta_{ij}\epsilon^j$$

(b) Each x_{ij}, y_{ij}, z_{ij}, δ_j, α_{ij}, β_{ij} is uniquely determined,
(c) $x'_i = y_i = z_i = 0$ and $z'_i = c_i + y_i$ at $t = 0$,
(d) The series are convergent for $|\epsilon|$ sufficiently small,
(e) The whole solution and therefore each x_{ij}, y_{ij}, z_{ij}, is periodic with the period $2\pi/\nu$.

The coefficients of ϵ^0. The terms of the power series which are independent of ϵ are determined by a set of linear homogeneous differential equations easily found from (1) by making $\epsilon = 0$. Their solutions satisfying condition c are

$$\begin{aligned}
x_{i0} &= K_{i1}\cos\rho_1 t + K_{i2}\cos\omega t + K_{i3}\cosh\rho_2 t,\\
y_{i0} &= L_{i1}\sin\rho_1 t + L_{i2}\sin\omega t + L_{i3}\sinh\rho_2 t + L_{i4}t,\\
z_{i0} &= M_{i1}\sin\omega t + M_{i2}\sin\nu t.
\end{aligned}$$

Where K_{ij} and L_{3j} are uniquely determined in terms of L_{1j} by the equations

$$\begin{aligned}
&(\lambda^2 - \omega^2 - 2A_1)K_{1j} - 2B_1K_{3j} - 2\lambda\omega L_{1j} = 0,\\
&-2B_3K_{1j} + (\lambda^2 - \omega^2 - 2A_3)K_{3j} - 2\lambda\omega L_{3j} = 0,\\
&2\lambda\omega K_{1j} + (\lambda^2 - \omega^2 + A_1)L_{1j} + B_1L_{3j} = 0, \qquad j = 1,\ 2,\ 3.
\end{aligned}$$

Let the initial conditions be

$$x_i = \alpha_i, \qquad y'_i = \beta_i, \qquad z'_i = c_i + \gamma_i \text{ at } t = 0.$$

The α_i, β_i are related to the L_{1j} by the equations.

$$\left.\begin{aligned}
\alpha_i^{(0)} &= a_1L_{11} + a_3L_{12} + a_5L_{13},\\
\alpha_3^{(0)} &= b_1L_{11} + b_3L_{12} + b_5L_{13},\\
\beta_1^{(0)} &= \rho_1\sqrt{-1}\,L_{11} + \omega\sqrt{-1}\,L_{12} + \rho_2L_{13} + L_{14},\\
\beta_3^{(0)} &= c_1\rho_1\sqrt{-1}\,L_{11} + c_3\omega\sqrt{-1}\,L_{12} + c_5\rho_2L_{13} + c_7L_{14},
\end{aligned}\right\} \quad (2)$$

Where $\alpha_1^{(0)}$, $\alpha_3^{(0)}$, $\beta_1^{(0)}$ and $\beta_3^{(0)}$ are the terms independent of ϵ in the power series which give α_1, α_3, β_1 and β_3 in terms of ϵ.

The periodicity condition (d) demands that

$$L_{11} = L_{12} = L_{13} = L_{14} = 0.$$

Hence

$$\alpha_1^{(0)} = \alpha_3^{(0)} = \beta_1^{(0)} = \beta_3^{(0)} = 0.$$

The M_{ij} are related to the initial constants by the equations

$$\left.\begin{aligned} c_1 + \gamma_1 &= \omega M_{11} + \nu M_{12}, \\ c_3 &= \gamma_3 - \omega \frac{\omega^2 - A_1}{B_1} M_{11} + \nu \; M_{12} \end{aligned}\right\} \quad (3)$$

The periodicity condition (d) also requires that $M_{11} = 0$, hence

$$c_3 + \gamma_3 = \frac{\nu^2 - A_1}{B_1} \; (c_1 + \gamma_1)$$

and

$$M_{12} = \frac{c_1 + \gamma_1}{\nu}$$

The terms of the series which are independent of ϵ are

$$x_{i0} = y_{i0} = 0,$$

$$z_{10} = \frac{c_1 + \gamma_1}{\nu} \sin \nu t,$$

$$z_{30} = \frac{\nu^2 - A_1}{B_1} \; \frac{c_1 + \gamma_1}{\nu} \sin \nu t.$$

The coefficients of ϵ.—These are determined by the following equations whose solutions must be subjected to the conditions (a), (b), (c), (d) and (e):

$$\left.\begin{aligned}
&\frac{d^2x_{11}}{dt^2}-2\omega\frac{dy_{11}}{dt}-\{(\omega^2+2A_1)x_{11}+2B_1x_{31}\}=z^2_{10}\frac{\partial^3U}{\partial x_1\partial z_1{}^2}\\
&\qquad\qquad+z_{10}z_{30}\frac{\partial^3U}{\partial x_1\partial z_1\partial z_3}+z^2_{30}\frac{\partial^3U}{\partial x_1\partial z_3{}^2},\\
&\frac{d^2x_{31}}{dt^2}-2\omega\frac{dy_{31}}{dt}-\{2B_3x_{11}+(\omega^2+2A_3)x_{31}\}=z^2_{10}\frac{\partial^3U}{\partial x_3\partial z_1{}^2}\\
&\qquad\qquad+z_{10}z_{30}\frac{\partial^3U}{\partial x_3\partial z_1\partial z_3}+z^2_{30}\frac{\partial^3U}{\partial x_3\partial z_3{}^2},\\
&\frac{d^2y_{11}}{dt^2}+2\omega\frac{dx_{11}}{dt}-(\omega^2-A_1)y_{11}+B_1y_{31}=0,\\
&\frac{d^2y_{31}}{dt^2}+2\omega\frac{dx_{31}}{dt}+B_3y_{11}-(\omega^2-A_3)y_{31}=0,\\
&\frac{d^2z_{11}}{dt^2}+A_1z_{11}+B_1z_{31}=-2\delta_1\{A_1M_{12}\sin\nu t+B_1M_{32}\sin\nu t\},\\
&\frac{d^2z_{31}}{dt^2}+B_3z_{11}+A_3z_{31}=-2\delta_1\{B_3M_{12}\sin\nu t+A_3M_{32}\sin\nu t\}.
\end{aligned}\right\}\quad(4)$$

The first four equations are independent of the last two. Their left members are the same as the left members of the equations which determined x_{i0} and y_{i0}. The right members can be thrown in the form $A_{10}{}^{(1)}+A_{12}{}^{(1)}\cos 2\nu t$, $A_{30}{}^{(1)}+A_{32}{}^{(1)}\cos 2\nu t$, 0, 0 where the A's all carry the arbitrary $c_1+\gamma_1$ as a quadratic factor.

The solution of (53) consists of a complementary function plus a particular integral. The complementary function has the wrong period as in the article just preceding and hence must vanish.

The particular integrals are

$$x_{i1}=K_{i0}{}^{(1)}+K_{i2}{}^{(1)}\cos 2\nu t,\ y_{i1}=L_{i0}{}^{(1)}+L_{i2}{}^{(1)}\sin 2\nu t$$

where $K_{i0}{}^{(1)}$, $K_{i2}{}^{(1)}$, $L_{i0}{}^{(1)}$ and $L_{i2}{}^{(1)}$ are determined so as to satisfy the differential equations. Applying the initial conditions we find

$$\alpha_i{}^{(1)}=K_{i0}{}^{(1)}+K_{i2}{}^{(1)},\ \beta_i{}^{(1)}=2\nu L_{i2}{}^{(1)}\qquad i=1,\ 3.$$

In the last two equations of (4) the solution again consists of a complementary function plus a particular integral. But the right members have the same period as the complementary function. Therefore in order that the solution be periodic the coefficients of the right members must be zero. These coefficients are

$$-2\delta_1(A_1M_{12}+B_1M_{32}),\quad -2\delta_1(B_3M_{12}+A_3M_{32}).$$

From the relation $M_{32} = \frac{\nu^2 - A_1}{B_1} M_{12}$ it is clear that the second factor cannot vanish hence $\delta_1 = 0$. Since the right members vanish the particular integrals are zero. The initial conditions by (c) are $z_{i1} = z'_{i1} = 0$ at $t = 0$. Therefore the complementary function is zero and the solutions satisfying the conditions which have been imposed are

$$z_{i1} = 0.$$

The coefficients of ϵ^2. These coefficients are determined by the equations

$$\left.\begin{aligned}
&\frac{d^2x_{13}}{dt^2} - 2\omega\frac{dy_{12}}{dt} - \{(\omega^2 + 2A_1)x_{12} + 2B_1x_{32}\} = 0,\\
&\frac{d^2x_{32}}{dt^2} - 2\omega\frac{dy_{32}}{dt} - \{2B_3x_{12} + (\omega^2 + 2A_3)x_{32}\} = 0,\\
&\frac{d^2y_{12}}{dt^2} + 2\omega\frac{dx_{12}}{dt} - (\omega^2 - A_1)y_{12} + B_1y_{32} = 0,\\
&\frac{d^2y_{32}}{dt^2} + 2\omega\frac{dx_{32}}{dt} + B_3y_{12} - (\omega^2 - A_3)y_{32} = 0,\\
&\frac{d^2z_{12}}{dt^2} + A_1z_{12} + B_1z_{32}\\
&\qquad = -2\delta_2(A_1M_{12} + B_1M_{32})\sin\nu t + c_{11}^{(2)}\sin\nu t + c_{13}^{(2)}\sin 3\nu t,\\
&\frac{d^2z_{32}}{dt^2} + B_3z_{12} + A_3z_{32}\\
&\qquad = -2\delta_2(B_3M_{12} + A_3M_{32})\sin\nu t + c_{32}^{(2)}\sin\nu t + c_{33}^{(2)}\sin 3\nu t.
\end{aligned}\right\}(5)$$

The solution of the first four equations consists simply of the complementary function. It has the wrong period and must vanish thus determining $\alpha_1^{(2)}$, $\alpha_3^{(2)}$, $\beta_1^{(2)}$ and $\beta_3^{(2)}$ all of which are zero. Hence

$$x_{i2} = y_{i2} = 0,$$

are the solutions satisfying the conditions.

In the last two equations the coefficients of $\sin\nu t$ must vanish giving

$$-2\delta_1(A_1M_{12} + B_1M_{32}) + c_{11}^{(2)} = 0,$$
$$-2\delta_1(B_3M_{12} + A_3M_{32}) + c_{31}^{(2)} = 0.$$

Since $M_{32} = \frac{\nu^2 - A_1}{B_1} M_{12}$ these give

$$\delta_2 = \frac{c_{11}^{(2)}}{2\nu\, M_{12}}, \text{ and } \delta_2 = \frac{B_1c_{31}^{(2)}}{2\nu^2(\nu^2 - A_1)M_{12}}.$$

By direct computation one can verify that

$$c_{31}{}^{(2)} = \frac{\nu^2 - A_1}{B_1}\, c_{11}{}^{(2)}.$$

Hence the two values of δ_2 are the same. The right members of the last two equations of (5) will contain only sines of odd multiples of t. Therefore the complete solution is

$$z_{i2} = M_{i1} \sin \omega t + M_{i2} \sin \nu t + K_i{}^{(2)} \sin 3\nu t,$$

where the $K_i{}^{(2)}$ have been determined by the usual method for finding the particular integral. By the initial conditions

$$\omega M_{11} + \nu M_{12} + 3\nu K_1{}^{(2)} = 0,$$

$$\omega\, \frac{\omega^2 - A_1}{B_1}\, M_{11} + \nu\, \frac{\nu^2 - A_1}{B_1}\, M_{12} + 3\nu K_3{}^{(2)} = \gamma_3{}^{(2)}.$$

Hence setting $M_{11} = 0$ as the periodicity conditions require we get

$$\gamma_3{}^{(2)} = 3\nu K_3{}^{(2)} - 3\nu K_1{}^{(2)}\, \frac{\nu^2 - A_1}{B_1}\quad .$$

A direct determination of the K's shows that this value of $\gamma_3{}^{(2)}$ is not zero.

The Normal Form for the Z-Equations. In each step of the construction there will appear two determinations of δ_i. In order to show that these two values are always the same we make a transformation of the form

$$\left.\begin{aligned} z_1 &= v_1 + v_2 + v_3 + v_4, \\ z'_1 &= \omega\sqrt{-1}(v_1 - v_2) + \nu\sqrt{-1}(v_3 - v_4), \\ z_3 &= d_1(v_1 + v_2) + d_2(v_3 + v_4), \\ z'_3 &= \omega\sqrt{-1}\, d_1(v_1 - v_2) + \nu\sqrt{-1}\, d_2(v_3 - v_4). \end{aligned}\right\} \qquad (6)$$

In the normal variables the Z-equations of motion become

$$\left.\begin{aligned} v'_1 &= (1 + \delta)\omega\sqrt{-1}\, v_1 + (1 + \delta)\epsilon\{C_1R_1 + C_3R_3\}, \\ v'_2 &= -(1 + \delta)\omega\sqrt{-1}\, v_1 + (1 + \delta)\epsilon\{-C_1R_1 - C_3R_3\}, \\ v'_3 &= (1 + \delta)\nu\sqrt{-1}\, v_3 + (1 + \delta)\epsilon\{-C_1R_1 + C_3R_3\}, \\ v'_4 &= -(1 + \delta)\nu\sqrt{-1}\, v_4 + (1 + \delta)\epsilon\{C_1R_1 - C_3R_3\}, \end{aligned}\right\} \qquad (7)$$

Where R_1 and R_3 are the right members of the original Z-equations. It is not necessary to change them into v's.

If we had made the previous part of the construction in the new variables it is clear that the coefficient of $e^{\nu\sqrt{-1}\, t}$ is the same in the v_3 and v_4 equations. Hence the two determinations of δ_2 are obviously the same.

The general term. Suppose that the

$$x_{i0}, \cdots x_{in}, \quad y_{i0}, \cdots y_{in}, \; z_{i0}, \cdots z_{in}, \quad \delta_1, \cdots \delta_{n-1},$$

have been determined and have the following properties:

(1) $x_{i2k} = y_{i2k} = z_{i2k+1} = \delta_{2k+1} = 0, \; \delta_{2k} \neq 0.$
(2) x_{i2k+1} are cosines of even multiples of νt, the highest being $2(k+1)$.
(3) y_{i2k+1} are sines of even multiples of νt, the highest being $2(k+1)$.
(4) z_{i2k} are sines of odd multiples of νt, the highest being $2k+1$.

It is possible to make an induction showing that these properties persist. This was done by Moulton in his Oscillating Satellite paper, for the case of an infinitesimal body. The details are so closely related to his work that it is not necessary to repeat it.

The character of the orbits. In the generating solutions $x_i = y_i = 0$, $z_i = M_i \sin \nu t$ there are apparently two constants M_1 and M_3 but $M_3 = \frac{\nu^2 - A_1}{B_1} M_1$ so that there is but one independent constant. The forms of the solutions show that $c_1 + \gamma_1$ enters as a factor of each x_{in}, y_{in} and z_{in} but to a power one greater than ϵ. Therefore, if we replace $(c_1 + \gamma_1)\epsilon$ by ϵ we get

$$x_i = (c_1 + \gamma_1) \sum_{j=0}^{\infty} x_{ij}\epsilon^j, \quad y_i = (c_1 + \gamma_1) \sum_{j=0}^{\infty} y_{ij}\epsilon^j, \quad z_i = (c_1 + \gamma_1) \sum_{j=0}^{\infty} z_{ij}\epsilon^j.$$

The coordinates in the orbits are given by

$$x'_i = \sum_{j=0}^{\infty} x_{ij}\epsilon^j, \quad y'_i = \sum_{j=0}^{\infty} y_{ij}\epsilon^j, \quad z'_i = \sum_{j=0}^{\infty} z_{ij}\epsilon^j,$$

where the primes denote the rectangular coordinates in rotating space. The exact forms of x'_i, y'_i, z'_i as far as the second power of ϵ' are

$$\left.\begin{aligned} x'_i &= \left\{K_{i0}^{(1)} + K_{i2}^{(1)} \cos \frac{2\nu t}{1+\delta}\right\}\epsilon'^2 + \cdots, \\ y'_i &= L_{i2}^{(1)} \sin \frac{2\nu t}{1+\delta} \, \epsilon'^2 + \cdots, \\ z'_i &= M_{i1}^{(1)} \sin \frac{\nu t}{1+\delta} \, \epsilon' + \cdots, \end{aligned}\right\} \qquad (8)$$

The equations of the projections of the curves on the plane is found by eliminating t between the first two equations. The result is

$$\left\{ \frac{x'_i - K_{i0}^{(1)}\epsilon'^2}{K_{i2}^{(1)}\epsilon'^2} \right\}^2 + \left\{ \frac{y'_i}{L_{i2}^{(1)}\epsilon'^2} \right\}^2 = 1$$

which are the equations of ellipses whose centers are on the x-axis.

The projections on the x', z' are the parabolas

$$x'_i = \{K_{i0}^{(1)} + K_{i2}^{(1)}(1 - \frac{2z'^2}{(M_{i1}^{(1)})^2\epsilon'^2})\}\epsilon'^2.$$

The projections of the y', z' plane are

$$y'_i = 2L_{i2}^{(1)} \frac{z'_i}{M_{i1}^{(1)}\epsilon} \sqrt{1 - \frac{z_i'^2}{(M_{i1}^{(1)}\epsilon)^2}}$$

The orbits are therefore curves shaped like the figure 8. It can be proved from equations that the parabolas are opening towards each other and that the values of y_i for small values of t are negative. The orbits are therefore retrograde. These projections show their general shape with reference to the rotating axes. The orbit of the middle body is always determined by the center of gravity equations.

Orbits with the period $2\pi/\omega$. The properties of the solutions which were established in the existence proof are:

(a) each x_{ij}, y_{ij}, z_{ij}, δ_j is uniquely determined.
(b) $x'_i = y_i = z_i = 0$ and $z'_i = c_i + \gamma_i$ at $t = 0$.
(c) The series are convergent for $0 < |\epsilon| < \epsilon_0$, $0 < t < T$.
(d) The whole solution and therefore each x_{ij}, y_{ij}, z_{ij} is periodic with period $2\pi/\omega$.

Applying these properties we find the coefficient of ϵ^0 to be

$$x_{i0} = K_{i1}^{(0)} \cos \omega t, \quad y_{i0} = L_{i1}^{(0)} \sin \omega t, \quad z_{i0} = M_{i1}^{(0)} \sin \omega t$$

since by the condition (d) the coefficients of $\cos \rho_1 t$, $\cosh \rho_2 t$, $\cos \nu t$ and of t must vanish. That is, we have

$$L_{i1} = L_{i3} = L_{i4} = M_{i3} = 0\ .$$

These equations are no longer homogeneous as in case 1 for the initial conditions are

$$x_{i0} = a_i + \alpha_i^{(0)}, \quad y'_{i0} = b_i + \beta_i^{(0)}, \quad z'_{i0} = c_i + \gamma_i^{(0)}.$$

The first three equations have a determinant different from zero. Therefore it is possible to solve for three of the initial constants in terms of the fourth, which was taken arbitrarly, and without loss of generality can be made equal to zero. The equation $M_{13} = 0$ gives γ_3 uniquely in terms of γ_1 which is so far undetermined. It will be necessary to determine γ_1 along with δ so that the solutions at the successive steps shall be periodic.

The coefficients of ϵ are determined by the differential equations.

$$\left.\begin{aligned} &x''_{i1} - 2\omega y'_{i1} - (\omega^2 + 2A_i)x_{i1} - 2B_i x_{k1} \\ &\qquad = 2\delta_1\{(\omega^2 + 2A_i)x_{i0} + 2B_i x_{k0}\} + A_{i0}^{(1)} + A_{i2}^{(1)} \cos 2\omega t, \\ &y''_{i1} + 2\omega y'_{i1} - (\omega^2 - A_i)y_{i1} + B_i y_{k1} \\ &\qquad = 2\delta_1\{\omega^2 - A_i)y_{i0} - B_i y_{k0}\} + B_{i0}^{(1)} + B_{i2}^{(1)} \sin 2\omega t, \\ &z''_{i1} + A_i z_{i1} + B_i z_{k1} \\ &\qquad = -2\delta_1(A_i z_{i0} + B_i z_{k0}) - \gamma_1^{(1)}(A_i z_{i0}^{(1)} + B_i z_{k0}^{(1)}) \\ &\qquad\qquad + C_{i0}^{(1)} + C_{i2}^{(1)} \sin 2\omega t, \quad (i = 1, 3, k = 3, 1). \end{aligned}\right\} \tag{9}$$

The superscript notation here means that the part of z_{i0} going with the first power of ϵ in γ_1 is taken. In these six equations the first terms of the right members have the same period as the solutions of the left members set equal to zero. In order that the solutions shall be periodic we must have

$$\gamma_1^{(1)} = \delta_1 = 0.$$

Then the solutions become

$$\left.\begin{aligned} x_{i1} &= A_{i0}^{(1)} + A_{i1}^{(1)} \cos \omega t + A_{i2}^{(1)} \cos 2\omega t, \\ y_{i1} &= B_{i0}^{(1)} + B_{i1}^{(1)} \sin \omega t + B_{i2}^{(1)} \sin 2\omega t, \\ z_{i1} &= C_{i0}^{(1)} + C_{i1}^{(1)} \sin \omega t + C_{i2}^{(1)} \sin 2\omega t, \end{aligned}\right\} \tag{10}$$

where the $A_{ij}^{(1)}$, $B_{ij}^{(1)}$, $C_{ij}^{(1)}$ are determined so as to satisfy the differential equations.

In the computation of the coefficients of ϵ^2 the equations which determine $\gamma_1^{(2)}$ and δ_2 are no longer homogeneous as above. Further, there are six linear equations which these two quantities must satisfy in order that the solutions shall be periodic. In order to show that these six equations reduce to two independent ones it is convenient to transform to the normal variables for the remainder of the construction.

The equations of motion become in the new variables.*

---

* See equations (22) *Am. Journal of Math.*, Apr. 1923, p. 101.

$$\left.\begin{aligned}
u'_1 &= (1+\delta)\phi\sqrt{-1}\,u_1 + (1+\delta)A_{11}\epsilon\{\textstyle\sum x_k y_j \dfrac{\partial^3 U}{\partial y_1 \partial y_j \partial x_k} + \epsilon(\ )\cdots\} \\
&\qquad + (1+\delta)A_{12}\epsilon\{\textstyle\sum x_k y_j \dfrac{\partial^3 U}{\partial y_3 \partial y_j \partial x_k} + \epsilon(\ )\cdots\} \\
&+ (1+\delta)A_{13}\epsilon\{\textstyle\sum x_k x_j \dfrac{\partial^3 U}{\partial x_1 \partial x_k \partial x_j} + \epsilon(\ )\cdots \sum y_k y_j \dfrac{\partial^3 U}{\partial x_1 \partial y_k \partial y_j} + \epsilon(\ )\cdots \\
&+ \textstyle\sum z_k z_j \dfrac{\partial^3 U}{\partial x_1 \partial z_k \partial z_j} + \epsilon(\ )\cdots\} + (1+\delta)A_{14}\epsilon\{\sum x_k x_j \dfrac{\partial^3 U}{\partial x_3 \partial x_k \partial x_j} + \epsilon(\ )\cdots \\
&+ \textstyle\sum y_k y_j \dfrac{\partial^3 U}{\partial x_3 \partial y_k \partial y_j} + \epsilon(\ )\cdots + \sum z_k z_j \dfrac{\partial^3 U}{\partial x_3 \partial z_k \partial z_j} + \epsilon(\ )\cdots\}, \\
u'_2 &= -(1+\delta)\phi_1\sqrt{-1}\,u_2 + (1+\delta)\epsilon \\
&\qquad \{-A_{11}(\ ) - A_{12}(\ ) - A_{13}(\ ) - A_{14}(\ )\}, \\
u'_3 &= (1+\delta)\omega\sqrt{-1}\,u_3 + (1+\delta)\epsilon \\
&\qquad \{B_{11}(\ ) + B_{12}(\ ) + B_{13}(\ ) + B_{14}(\ )\}, \\
u'_4 &= -(1+\delta)\omega\sqrt{-1}\,u_4 + (1+\delta)\epsilon \\
&\qquad \{-B_{11}(\ ) - B_{12}(\ ) - B_{13}(\ ) - B_{14}(\ )\}, \\
u'_5 &= (1+\delta)\phi_2\,u_5 + (1+\delta)\epsilon \\
&\qquad \{C_{11}(\ ) + C_{12}(\ ) + C_{13}(\ ) + C_{14}(\ )\}, \\
u'_6 &= -(1+\delta)\phi_2\,u_6 + (1+\delta)\epsilon \\
&\qquad \{-C_{11}(\ ) - C_{12}(\ ) - C_{13}(\ ) - C_{14}(\ )\}, \\
u'_7 &= \qquad (1+\ )\delta\epsilon\,\{D_{11}(\ ) + D_{12}(\ ) \qquad\qquad \}, \\
u_8' &= \qquad u_7 + (1+\delta)\epsilon \\
&\qquad \{D_{11}(\ ) + D_{12}(\ ) - D_{13}(\ ) - D_{14}(\ )\}, \\
z_1'' + (1+\delta)^2(A_1 z_1 + B_1 z_3) &= (1+\delta)^2\epsilon \textstyle\sum x_k z_j \dfrac{\partial^3 U}{\partial z_1 \partial x_k \partial z_j} \cdots, \\
z_3'' + (1+\delta)^2(B_3 z_1 + A_3 z_3) &= (1+\delta)^2\epsilon \textstyle\sum x_k z_j \dfrac{\partial^3 U}{\partial z_3 \partial x_k \partial z_j} \cdots,
\end{aligned}\right\}\quad (11)$$

where the parentheses denote the coefficients of A_{11}, A_{12}, A_{13}, A_{14} in the first equation.

For the sake of clearness we will compute again the coefficients of ϵ^0 and ϵ. Let the initial conditions be

$$u_i = a_i, \quad (i = 1, 2, 5, 6, 7, 8), \quad u_3 = a + a_3, \quad u_4 = a + a_4,$$
$$z'_i = c_i + \gamma_i, \qquad c_i = \omega M_i.$$

Therefore the coefficients of ϵ^0 are

$$u_{30} = (a + \alpha_3)e^{\omega\sqrt{-1}\,t}, \quad u_{40} = (a + \alpha_4)e^{\omega\sqrt{-1}\,t},$$
$$z_{i0} = M_{i0} \sin \omega t. \qquad u_{i0} = 0, \; (i = 1, 2, 5, 6, 7, 8),$$

By the orthogonality conditions $\alpha_3 = \alpha_4$ and therefore

$$\left.\begin{array}{ll} x_{10} = 2a_1(a + \alpha) \cos \omega t, & x_{30} = 2b_1(a + \alpha) \cos \omega t, \\ y_{10} = 2(a + \alpha) \sin \omega t, & y_{30} = 2c_1(a + \alpha \sin \omega t, \\ \multicolumn{2}{c}{z_{i0} = M_{i0} \sin \omega t.} \end{array}\right\} \qquad (12)$$

The coefficients of ϵ are determined by a set of equations like (11) in which all the terms containing ϵ to a higher power than the first are omitted and the values of x_{i0}, y_{i0}, z_{i0} are substituted from (12). It is easily seen that the periodicity conditions are satisfied by $\delta_1 = \gamma_1^{(1)} = 0$. The solutions are

$$\left.\begin{array}{ll} u_{11} = & a_1^{(1)} + a_2^{(1)} \sin 2\omega t + a_3^{(1)} \cos 2\omega t, \\ u_{21} = & a_1^{(1)} - a_2^{(1)} \sin 2\omega t + a_3^{(1)} \cos 2\omega t, \\ u_{31} = & b_0^{(1)} e^{\omega\sqrt{-1}\,t} + b_1^{(1)} + b_2^{(1)} \sin 2\omega t + b_3^{(1)} \cos 2\omega t, \\ u_{41} = & b_0^{(1)} e^{-\omega\sqrt{-1}\,t} + b_1^{(1)} - b_2^{(1)} \sin 2\omega t + b_3^{(1)} \cos 2\omega t, \\ u_{51} = & c_1^{(1)} + c_2^{(1)} \sin 2\omega t + c_3^{(1)} \cos 2\omega t, \\ u_{61} = & c_1^{(1)} - c_2^{(1)} \sin 2\omega t + c_3^{(1)} \cos 2\omega t, \\ u_{71} = & d_1^{(1)} + d_2^{(1)} \sin 2\omega t + d_3^{(1)} \cos 2\omega t, \\ u_{81} = & d_1^{(1)} - d_2^{(1)} \sin 2\omega t + d_3^{(1)} \cos 2\omega t, \\ \multicolumn{2}{l}{z_{i1} = M_{i1}^{(1)} \sin \omega t + c_{i1}^{(1)} + c_{i2}^{(1)} \sin \omega t.} \end{array}\right\} \qquad (13)$$

The two arbitrary constants of integration, $b_0^{(1)}$ and $b^{(1)}$ are determined by the orthogonality conditions so that $b_0^{(1)} = b^{(1)} = b/2$ where b is a known quantity. Therefore in the x_i, y_i, z_i variables we have

$$x_{i1} = A_{i0}^{(1)} + A_{i1}^{(1)} \cos \omega t + A_{i2}^{(1)} \cos 2\omega t,$$
$$y_{i1} = B_{i0}^{(1)} + B_{i1}^{(1)} \sin \omega t + B_{i2}^{(1)} \sin 2\omega t,$$
$$z_{i1} = C_{i0}^{(1)} + C_{i1}^{(1)} \sin \omega t + C_{i2}^{(1)} \sin 2\omega t.$$

The character of the differential equations together with the initial conditions shows that x_i will always be a cosine series, y_i a sine series, z_i a sine series, provided $\gamma_1^{(n)}\delta_n$ can be determined so that the solutions are periodic. We will prove that there are just enough independent equations to determine $\gamma_1^{(n)}$ and δ_n uniquely. The equations defining the coefficients of ϵ^n are

$$\left.\begin{array}{l}
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
u'_{3n} = \omega\sqrt{-1}\,u_{3n} + \delta_n\omega\sqrt{-1}\,u_{30} + A_{30}^{(n)} + A_{31}^{(n)}\sin\omega t \\
\qquad + A_{32}^{(n)}\sin 2\omega t + \cdots + b_{31}^{(n)}\cos\omega t + b_{32}^{(n)}\cos 2\omega t + \cdots, \\
u'_{4n} = -\omega\sqrt{-1}\,u_{4n} + \delta_n\omega\sqrt{-1}\,u_{30} + A_{30}^{(n)} + A_{31}^{(n)}\sin\omega t \\
\qquad + A_{32}^{(n)}\sin 2\omega t + \cdots - b_{31}^{(n)}\cos\omega t - b_{32}^{(n)}\cos 2\omega t + \cdots, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
z''_{in} = -A_1z_{1n} - B_1z_{3n} - 2\delta_n(A_1z_{10} + B_1z_{30}) - \gamma_1^{(n)}(A_1z_{10}^{(n)} + B_1z_{30}^{(n)}) \\
\qquad\qquad + c_{11}^{(n)} + c_{12}^{(n)}\sin\omega t + c_{13}^{(n)}\sin 2\omega t + \cdots \\
z''_{3n} = -B_3z_n - A_3z_{3n} - 2\delta_n(B_3z_{10} + A_3z_{30}) - \gamma_1^{(n)}(B_3z_{10}^{(n)} + A_3z_{30}^{(n)} \\
\qquad\qquad + c_{31}^{(n)} + c_{32}^{(n)}\sin\omega t + c_{33}^{(n)}\sin 2\omega t + \cdots
\end{array}\right\}(14)$$

where the equations omitted are already in such a form that the periodicity conditions are satisfied. The condition that the first equation of (14) shall have a periodic solution is

$$\omega\sqrt{-1}\,(a + \alpha)\delta_n + \frac{a_{31}^{(n)}}{2\sqrt{-1}} + \frac{b_{31}^{(n)}}{2} = 0$$

since the coefficient of $e^{\omega\sqrt{-1}\,t}$ must vanish. In the second equation the coefficient of $e^{-\omega\sqrt{-1}\,t}$ must vanish, which gives exactly the same equation as the above. This equation uniquely determines δ_n.

In order that the last two equations shall give periodic solutions we must have

$$\begin{aligned}
2\delta_n(A_1M_{10} + B_1M_{30}) + \gamma_1^{(n)}(A_1M_{10} + B_1M_{30}) &= \phi_{12}^{(n)}, \\
2\delta_n(B_3M_{10} + A_3M_{30}) + \gamma_1^{(n)}(B_3M_{10} + A_3M_{30}) &= \phi_{32}^{(n)}.
\end{aligned}$$

These two equations can be shown to be identical by reducing the z_i equations to the normal form as in the last article. Since δ_n is already known, $\gamma_1^{(n)}$ is uniquely determined by either of these equations.

In the generating solutions there are two independent arbitrary numbers L_1, M_1. It is no restriction on their independence to put $M_1 = CL_1$. Then in the solutions constructed above, L_1 enters as a factor of each x_{ij}, y_{ij}, z_{ij} but to a power one greater than ϵ. Therefore in the expressions for the coordinates, L_1 can be absorbed into ϵ. The C is left arbitrary.

We have investigated the orbits with the period $2\pi/\omega$ under general initial conditions, and under orthogonal initial conditions. Let us now consider the possibility of orbits with period $2\pi/\omega$ such that the coordinates are in the following proportion

$$\frac{x_1}{x_3} = \frac{y_1}{y_3} = \frac{z_1}{z_3} = \text{a constant.}$$

If we make this substitution in equations (2), they reduce to the two-body problem and are completely integrable.*

When the above proportion is satisfied the bodies all move in ellipses and remain in a straight line. The plane of motion may have any inclination whatever, which accounts for the arbitrary C left in the construction above.

Orbits with Period $2\pi/\rho_1$. We have already proved that these orbits lie wholly in the xy-plane. The method of constructing them is very similar to that of case 2 with the z-equations omitted. It is again necessary to transform to the normal form in order to show that the two determinations of δ_n are identical.

The geometrical character of the orbits is described by the following statements:

(1) The orbits are symmetrical with respect to the x-axis geometrically and the time.

(2) Omitting powers of ϵ beyond the first, the orbits are ellipses with centers on the x-axis.

Orbits in which ρ_1 *and* ν *are commensurable.* These are of the same character as the orbits with period $2\pi/\omega$. The same difficulties of construction arise and may be handled in a similar fashion. The orbits are symmetrical with respect to the x-axis geometrically and in the time. In case (2) we showed that the orbits lie wholly in one plane with an arbitrary inclination but in this case there does not seem to be such a simple interpretation of the constants. The projections of the orbits on the xy-plane; neglecting powers of ϵ beyond the first, are ellipses with centers on the x-axis. The period is the least common multiple of $2\pi/\rho_1$ and $2\pi/\nu$.

---

* Lagrange, *Collected Works,* VI, pp. 230-270.

Correction.

On page 223 of the current volume of this JOURNAL, the first parametric equation for curves (9) and (10) should read

$$x_0 = t^7 + at.$$